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On I-Vague Products

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ABSTRACT: The notions of I-vague product of groups with membership and non-membership functions taking values in a complete involuntary dually residuated lattice ordered semigroup are introduced. This generalizes the notions with truth values in a complete Boolean algebra as well as those usual vague sets whose membership and non-membership functions taking values in the unit interval [0, 1]. We prove that if the complete involuntary dually residuated lattice ordered semigroup is infinitely meet distributive, then the set of all I-vague normal groups of a group with I-vague product forms a semilattice.

Key words/phrases: Involutary dually residuated lattice ordered semigroup, I-vague group, I-vague normal group, I-vague product and I-vague set

INTRODUCTION

Ramakrishna and Eswarlal (2008) studied Boolean vague sets where the vague set of the universe X is defined by the pair of functions (t_A, f_A) where t_A and f_A are mappings from a set X into a Boolean algebra A satisfying the condition $t_A(x) \leq f_A(x)'$ for all $x \in$X where $f_A(x)'$ is the complement of $f_A(x)$in the Boolean algebra A. Swamy (1965a, 1965b, 1966) introduced the concept of a Dually Residuated Lattice Ordered Semigroup (in short DRL -semigroup) which is a common abstraction of Boolean algebras and lattice ordered semigroups. The subclass of DRL -semigroups which are bounded and involuntary (i.e having 0 as least, 1 as greatest and satisfying $1 - (1 - x) = x$) which is categorically equivalent to the class of MV-algebras of Chang (1958) and well -studied offer a natural generalization of the closed unit interval [0, 1] of real numbers as well as Boolean algebras. Thus, the study of vague sets (t_A, f_A) with values in an involuntary DRL – semigroup (Zelalem Teshome, 2010) promises a unified study of real valued vague sets and also those Boolean valued vague sets.

Ramakrishna (2008) studied on a product of vague groups by introducing the concept of vague product. In this paper, using the definition of I-vague groups in (Zelalem Teshome, 2011a) and I-vague normal groups in (Zelalem Teshome, 2011b), we define and study I-vague product where I is a complete involuntary DRL - semigroup which generalizes the work of Ramakrishna (2008). Throughout this paper, we shall denote the identity element of a group G by e.

Preliminaries

Definition 2.1: A system (A, +, ≤, -) is called a dually residuated lattice ordered semigroup(in short DRL -semigroup) if and only if
i) (A, +) is a commutative semigroup with zero "0";
ii) (A, ≤) is a lattice such that
$a + (b \cup c) = (a + b) \cup (a + c)$ and $a + (b \cap c) = (a + b) \cap (a + c)$ for all $a, b, c \in$ A.
iii) Given $a, b \in$ A, there exists a least x in A such that $b + x \geq a$, and we denote this x by $a - b$ (for a given a, b this x is uniquely determined);
iv) $(a - b) \cup 0 + b \leq a \cup b$ for all $a, b \in$ A;
v) $a - a \geq 0$ for all $a \in$A.

Theorem 2.2: Any DRL-semigroup is a distributive lattice.

Definition 2.3: A DRL -semigroup A is said to be involuntary if there is an element $1(\neq 0)$(0 is the identity w.r.t. +) such that
i) $a + (1 - a) = 1 + 1$;
ii) $1 - (1 - a) = a$ for all $a \in$A.

Theorem 2.4: In DRL -semigroup with 1, 1 is unique.

Theorem 2.5: If a DRL -semigroup contains a least element x, then $x = 0$. Dually, if a DRL - semigroup with 1 contains a largest element α, then $\alpha = 1$.

Throughout this paper let I = (I, +, – , ∨, ∧, *0*, *1*) be a dually residuated lattice ordered semigroup satisfying $1-(1-a)=a$ for all $a\in$ I.

Lemma 2.6 Let *1* be the largest element of I. Then for $a, b\in$ I

i) $a+(1-a)=1$.

ii) $1-a=1-b \Leftrightarrow a=b$.

iii) $1-(a\vee b)=(1-a)\wedge(1-b)$.

Lemma 2.7: Let I be complete. If $a_\alpha\in$ I for every $\alpha\in\Delta$, then

i) $1-\bigvee_{\alpha\in\Delta}a_\alpha=\bigwedge_{\alpha\in\Delta}(1-a_\alpha)$.

ii) $1-\bigwedge_{\alpha\in\Delta}a_\alpha=\bigvee_{\alpha\in\Delta}(1-a_\alpha)$.

Definition 2.8: An I-vague set A of a non-empty set G is a pair (t_A, f_A) where t_A: G→I and f_A: G→I with $t_A(x)\le 1-f_A(x)$ for all $x\in$ G.

Definition 2.9: The interval $[t_A(x), 1-f_A(x)]$ is called the I-vague value of $x\in$ G and denoted by $V_A(x)$.

Definition 2.10: Let $B_1=[a_1, b_1]$ and $B_2=[a_2, b_2]$ be I-vague values. We say $B_1\ge B_2$ if and only if $a_1\ge a_2$ and $b_1\ge b_2$.

Definition 2.11: Let A = (t_A, f_A) and B = (t_B, f_B) be I-vague sets of a set G. Ais said to be contained in B written A ⊆B if and only if $t_A(x)\le t_B(x)$ and $f_A(x)\ge f_B(x)$ for all $x\in$G. A is said to be equal to B written as A = B if and only if A ⊆ B and B⊆ A.

Definition 2.12: Let A = (t_A, f_A) and B = (t_B, f_B) be I-vague sets of a set G.

i) Their union A∪B is defined as

$A\cup B=(t_{A\cup B}, f_{A\cup B})$ where $t_{A\cup B}(x)=t_A(x)\vee t_B(x)$ and $f_{A\cup B}(x)=f_A(x)\wedge f_B(x)$ for all $x\in$G.

ii) Their intersection A∩B is defined as

$A\cap B=(t_{A\cap B}, f_{A\cap B})$ where $t_{A\cap B}(x)=t_A(x)\wedge t_B(x)$ and $f_{A\cap B}(x)=f_A(x)\vee f_B(x)$ for all $x\in$G.

Definition 2.13: Let $B_1=[a_1, b_1]$ and $B_2=[a_2, b_2]$ be I-vague values. Then

i) isup$\{B_1, B_2\}$= [sup$\{a_1, a_2\}$, sup$\{b_1, b_2\}$].

ii) iinf$\{B_1, B_2\}$= [inf$\{a_1, a_2\}$, inf$\{b_1, b_2\}$].

Lemma 2.14: Let A and B be I-vague sets of a set G. Then A∪B and A∩B are also I-vague sets of G.

Let $x\in$G. From the definition of A∪B and A∩B we have

i) $V_{A\cup B}(x)$ = isup $\{V_A(x), V_B(x)\}$;

ii) $V_{A\cap B}(x)$ = iinf$\{V_A(x), V_B(x)\}$.

Definition 2.15: Let I be complete and $\{A_i=(t_{A_i},f_{A_i})$: $i\in\Delta\}$ be a non empty family of I-vague sets of G. Then for each $x\in$G.

i) isup$\{V_{A_i}(x)$: $i\in\Delta\}$ = $[\bigvee_{i\in\Delta}t_{A_i}(x), \bigvee_{i\in\Delta}(1-f_{A_i}(x))]$

ii) iinf$\{V_{A_i}(x)$: $i\in\Delta\}$ = $[\bigwedge_{i\in\Delta}t_{A_i}(x) ,\bigwedge_{i\in\Delta}(1-f_{A_i}(x))]$

Definition 2.16: Let G be a group. An I-vague set A of a group G is called an I-vague group of G if

i) $V_A(xy)\ge$iinf$\{V_A(x), V_A(y)\}$ for all $x, y\in$G.

ii) $V_A(x^{-1})\ge V_A(x)$ for all $x\in$G.

Lemma 2.17: If A is an I-vague group of a group G, then $V_A(x)=V_A(x^{-1})$ for all $x\in$G.

Lemma 2.18: If A is an I-vague group of a group G, then $V_A(e)\ge V_A(x)$ for all $x\in$G.

Lemma 2.19: A necessary and sufficient condition for an I-vague set A of a group G is an I-vague group of G is that $V_A(xy^{-1})\ge$iinf$\{V_A(x), V_A(y)\}$ for all $x, y\in$ G.

Lemma 2.20: If A and B are I-vague groups of a group G, then A∩B is also an I-vague group of G.

Definition 2.21: Let G be a group. An I-vague group A of G is called an I-vague normal group of G if for all $x, y\in$G, $V_A(xy)=V_A(yx)$.

Lemma 2.22: Let A be an I-vague group of a group G. A is an I-vague normal group of G if and only if $V_A(x)=V_A(yxy^{-1})$ for all $x, y\in$G.

Theorem 2.23: If A and B are I-vague normal groups of a group G, then A∩B is also an I-vague normal group of G.

I-Vague Products

Throughout this section I is complete.

Definition 3.1: Let A = (t_A, f_A) and B = (t_B, f_B)be I-vague sets of a group G. Then the product of A and B, denoted A o B = (t_{AoB}, f_{AoB}) is defined as $t_{AoB}(x)$ = sup{inf$\{t_A(y), t_B(z)\}$: $y, z\in$G, $x=yz\}$ = $\bigvee_{x=yz}[t_A(y)\wedge t_B(z)]$ and

$f_{AoB}(x)$ = inf{sup$\{f_A(y), f_B(z)\}$: $y,z\in$G, $x=yz\}$ = $\bigwedge_{x=yz}[f_A(y)\vee f_B(z)]$.

Since $x=xe=ex$ for all $x\in$G, $t_{AoB}$and $f_{AoB}$are defined for all $x\in$G.

Lemma 3.2: If A and B are I-vague sets of a group G, then A o B is also an I-vague set of G.

Proof: Let A = (t_A, f_A) and B = (t_B, f_B)be I-vague sets of G. Let x∈G. Then $t_A(x) \leq 1 - f_A(x)$and $t_B(x) \leq 1 - f_B(x)$.

$t_{AoB}(x) = \vee_{x=yz}[t_A(y) \wedge t_B(z)]$

$\leq \vee_{x=yz}[(1 - f_A(y)) \wedge (1 - f_B(z))]$

$= \vee_{x=yz}[1 - (f_A(y) \vee f_B(z)]$

by lemma 2.6 (iii)

$= 1 - \wedge_{x=yz}(f_A(y) \vee f_B(z))$

$= 1 - f_{AoB}(x)$.

Thus $t_{AoB}(x) \leq 1 - f_{AoB}(x)$ for all x∈G.

Hence the lemma follows.

Lemma 3.3: If A and B are I-vague sets of a group G, then

$V_{A\,o\,B}(x)$ = isup{iinf $\{V_A(y), V_B(z)\}$: y, z∈G, $x = yz$} for each x∈G.

Proof: Let A = (t_A, f_A) and B = (t_B, f_B)be I-vague sets of a group G.

Let x∈G. Then

$t_{AoB}(x) = \vee_{x=yz}[t_A(y) \wedge t_B(z)]$ and

$f_{AoB}(x) = \wedge_{x=yz}[f_A(y) \vee f_B(z)]$ where y, z∈G.

$V_{A\,o\,B}(x) = [t_{AoB}(x), 1 - f_{AoB}(x)]$

$= [\vee_{x=yz}(t_A(y) \wedge t_B(z)),$

$1 - \wedge_{x=yz}(f_A(y) \vee f_B(z))]$

$= [\vee_{x=yz}(t_A(y) \wedge t_B(z)),$

$$\bigvee_{x=yz} (1 - f_A(y)) \wedge (1 - f_B(z))]$$

$= \vee_{x=yz}[t_A(y) \wedge t_B(z), (1 - f_A(y)) \wedge (1 - f_B(z))]$

= isup{iinf $\{V_A(y), V_B(z)\}$: y, z∈G, $x = yz$}.

Thus $V_{A\,o\,B}(x)$ = isup{iinf$\{V_A(y), V_B(z)\}$: y, z∈G, $x = yz$} for each x∈G.

Example 3.4 : Let I = the positive divisors of 30 = {1, 2, 3, 5, 6, 10, 15, 30}

In which

$x \vee y$ = The least common multiple of x and y.

$x \wedge y$ = The greatest common divisor of x and y.

$x' = \frac{30}{x}$.

Then I = (I, ∨,∧,′ , 1, 30) is a Boolean algebra. Hence it is an involutary DRL-semigroup.

Consider the group G = (Z, +). Then H = (2Z, +) and K = (3Z, +) are subgroups of G. Define the I-vague groups A and B of G as follows:

$$V_A(x) = \begin{cases} [15, 30]\, if\, x \in H\,; \\ [5,10]\, otherwise. \end{cases}$$

and

$$V_B(x) = \begin{cases} [15, 30]\; if\, x \in K; \\ [5,10]\, otherwise. \end{cases}$$

Then

$V_{AoB}(x)$ = [15, 30] for all x∈ G

Corollary 3.5: If a group G is abelian and A and B are I-vague sets of G, then A o B = B o A.

Proof: Let x∈G. Then

$V_{A\,o\,B}(x)$

= isup{iinf$\{V_A(y), V_B(z)\}$: y, z∈G, $x = yz$}

= isup{iinf$\{V_B(z), V_A(y)\}$: y, z∈G, $x = zy$}

$= V_{B\,o\,A}(x)$.

Thus $V_{A\,o\,B}(x) = V_{B\,o\,A}(x)$ for each x∈ G.

Hence A o B = B o A.

Theorem 3.6: Let A and B be I-vague sets of a group G. Then

i) A ⊆A o B if and only if $V_A(e) \leq V_B(e)$.

ii) B ⊆ A o B if and only if $V_B(e) \leq V_A(e)$.

Proof: Let A and B be I-vague sets of G.

i) Suppose that A⊆A o B.

$V_{AoB}(e)$ = isup{iinf$\{V_A(x), V_B(x^{-1})\}$: x∈G}

= iinf$\{V_A(e), V_B(e)\}$

$\leq V_B(e)$by definition 2.10.

Hence $V_{AoB}(e) \leq V_B(e)$.

Since A ⊆A o B, $V_A(e) \leq V_{AoB}(e)$.

Therefore $V_A(e) \leq V_B(e)$.

Conversely, suppose that $V_A(e) \leq V_B(e)$. Now we prove that $V_A(x) \leq V_{AoB}(x)$for each x∈G.

$V_{A\,o\,B}(x)$ = isup{iinf$\{V_A(y), V_B(z)\}$: y, z ∈G, $x = yz$}

≥iinf$\{V_A(x), V_B(e)\}$

≥iinf$\{V_A(x), V_A(e)\}$

= $V_A(x)$ by lemma 2.18.

Thus $V_A(x) \leq V_{A\,o\,B}(x)$for each x∈ G.

Therefore A⊆A o B.

Hence (i) holds true.

ii) Suppose that B ⊆A o B.

B ⊆ A o B implies $V_B(e) \leq V_{AoB}(e)$.

$V_{AoB}(e)$ = isup{iinf$\{V_A(x), V_B(x^{-1})\}$: x∈G}

=iinf$\{V_A(e), V_B(e)\}$

$\leq V_A(e)$.

Hence $V_{AoB}(e) \leq V_B(e)$.

Since $V_B(e) \leq V_{AoB}(e)$ and $V_{AoB}(e) \leq V_A(e)$, it follows that $V_B(e) \leq V_A(e)$.

Conversely, suppose that $V_B(e) \leq V_A(e)$.

$V_{A\,o\,B}(x)$ = isup{iinf $\{V_A(y), V_B(z)\}$: y, z∈G, $x = yz$}

≥iinf $\{V_A(e), V_B(x)\}$

≥iinf $\{V_B(e), V_B(x)\}$

= $V_B(x)$.

Hence $V_B(x) \leq V_{A\,o\,B}(x)$for each x∈G.

Therefore B ⊆A o B. Thus (ii) holds true.

Hence the theorem follows.

Corollary 3.7: Let A and B be I-vague sets of a group G. Then $A \subseteq A \text{ o } B$ and $B \subseteq A \text{ o } B$ iff $V_A(e) = V_B(e)$.

Proof: Let A and B be I-vague sets of G. Suppose that $A \subseteq A \text{ o } B$ and $B \subseteq A \text{ o } B$. By theorem 3.6, $A \subseteq A \text{ o } B$ if and only if $V_A(e) \leq V_B(e)$. Moreover, $B \subseteq A \text{ o } B$ if and only if $V_B(e) \leq V_A(e)$. Therefore $A \subseteq A \text{ o } B$ and $B \subseteq A \text{ o } B$ iff $V_A(e) = V_B(e)$.

Lemma 3.8: If A is an I-vague group of a group G, then A o A = A.

Proof: Let $x \in G$. Since A is an I-vague group of a group G,
$V_A(x) = V_A(yz)$
$\geq \text{iinf}\{V_A(y), V_A(z)\}$ when ever $x = yz$ for $y, z \in G$.
It follows that $V_A(x) \geq \text{isup}\{\text{iinf}\{V_A(y), V_A(z)\}: y, z \in G, x = yz\}$.
Hence $V_A(x) \geq V_{A\text{o}A}(x)$. Thus $A \supseteq A \text{ o } A$.
$V_{A \text{ o } A}(x) = \text{isup}\{\text{iinf}\{V_A(y), V_A(z)\}: y, z \in G, x = yz\}$
$\geq \text{iinf}\{V_A(x), V_A(e)\}$
$= V_A(x)$.
Thus $V_{A \text{ o } A}(x) \geq V_A(x)$ for each $x \in G$.
Hence $A \text{ o } A \supseteq A$.
Therefore A o A = A.

Example 3.9: Let I be the unit interval [0, 1] of real numbers. Define $a \oplus b = \min\{1, a + b\}$. With the usual ordering $(I, \oplus, \leq, -)$ is an involutary DRL-semigroup.
Consider G = (Z, +) and H = (3Z, +). Let A be the I-vague group of G defined by

$$V_A(x) = \begin{cases} [1/2,\ 1] & if\, x \in H; \\ [0,\ 3/4] & otherwise. \end{cases}$$

Since A be the I-vague group of G, Ao A = A. Hence

$$V_{A\text{o}A}(x) = \begin{cases} [1/2,\ 1] & if\, x \in H; \\ [0,\ 3/4] & otherwise. \end{cases}$$

Theorem 3.10: Let A be an I-vague set of a group G. Then A is an I-vague group of G if

i) A o A = A;
ii) $V_A(x) = V_A(x^{-1})$ for each $x \in G$.

Proof: Suppose that A is an I-vague group of G. By lemma 3.8, A o A = A.
Moreover, $V_A(x) = V_A(x^{-1})$ for all $x \in G$.

Conversely, suppose that A o A = A and $V_A(x) = V_A(x^{-1})$ for each $x \in G$.
We prove that $V_A(xy) \geq \text{iinf}\{V_A(x), V_A(y)\}$ for all $x, y \in G$.
$V_A(xy) = V_{A \text{ o } A}(xy)$
$= \text{isup}\{\text{iinf}\{V_A(a), V_A(b)\}: a, b \in G, xy = ab\}$
$\geq \text{iinf}\{V_A(x), V_A(y)\}$ by taking $a = x$ and $b = y$.
Hence $V_A(xy) \geq \text{iinf}\{V_A(x), V_A(y)\}$ for all $x, y \in G$.
Therefore A is an I-vague group of G.
Hence the theorem follows.

Definition 3.11: Let A be an I-vague group of a group G. A is said to be the I-vague group of G generated by an I-vague set $B \subseteq A$ if A is the smallest I-vague group of G containing B.

Theorem 3.12: Let A and B be I-vague groups of a group G with $V_A(e) = V_B(e)$. If A o B is an I-vague group of G, then the I-vague product A o B is the I-vague group of G generated by $A \cup B$.

Proof: Suppose that A o B is an I-vague group of G and $V_A(e) = V_B(e)$. Since $V_A(e) = V_B(e)$, it follows that $A \subseteq A \text{ o } B$ and $B \subseteq A \text{ o } B$ by corollary 3.7. Therefore, A o B is an I-vague group of G containing both A and B. Let C be an I-vague group of G containing A and B. Then, $A \subseteq C$ and $B \subseteq C$.
Let $x \in G$. Then
$V_{A \text{ o } B}(x) = \text{isup}\{\text{iinf}\{V_A(y), V_B(z)\}: y, z \in G, x = yz\}$
$\leq \text{isup}\{\text{iinf}\{V_C(y), V_C(z)\}: y, z \in G, x = yz\}$
$= V_{C \text{ o } C}(x)$
$= V_C(x)$.
Thus $V_{A \text{ o } B}(x) \leq V_C(x)$ for each $x \in G$.
Therefore $A \text{ o } B \subseteq C$.
Hence the theorem follows.

Example 3.13: Let I be the unit interval [0, 1] of real numbers. Define $a \oplus b = \min\{1, a + b\}$. With the usual ordering $(I, \oplus, \leq, -)$ is an involutary DRL-semigroup. Consider G = (Z, +) and H = (3Z, +). Define the I-vague groups A and B of G as follows:

$$V_A(x) = \begin{cases} [1/2,\ 1] & if\, x \in H; \\ [0,\ 1/4] & otherwise. \end{cases} \quad \text{and}$$

$$V_B(x) = \begin{cases} [1/2,\ 1] & if\, x \in H; \\ [1/4,\ 1/3] & otherwise. \end{cases}$$

Hence

$$V_{A\text{o}B}(x) = \begin{cases} [1/2,\ 1] & if\, x \in H; \\ [1/4,\ 1/3] & otherwise. \end{cases}$$

A and B be I-vague groups of the group Z with $V_A(e) = V_B(e)$. Moreover, A o B is an I-vague group of G. Hence the I-vague product A o B is the I-vague group of G generated by $A \cup B$.

Theorem 3.14: Let A and B be I-vague groups of a group G. If A or B is an I-vague normal group of G, then A o B = B o A.

Proof: Let A and B be I-vague groups of G. Suppose that A is an I-vague normal group of G. We prove that $V_{A\,o\,B}(x)= V_{B\,o\,A}(x)$for each $x\in$G.
Let $x\in$G. Then
$V_{A\,o\,B}(x)$ = isup{iinf{$V_A(y)$, $V_B(z)$}: $y, z\in$G, $x = yz$}
= isup{iinf{$V_A(z^{-1}yz)$, $V_B(z)$}: $y, z\in$G, $x = yz$}
= isup{iinf{$V_B(z)$, $V_A(z^{-1}yz)$}: $y, z\in$G, $x = yz$ }
Set $z' = z^{-1}yz$. Then $zz' = yz$.
$V_{A\,o\,B}(x)$ = isup{iinf {$V_B(z)$, $V_A(z^{-1}yz)$}: $y, z\in$G, $x = yz$}
= isup{iinf{$V_B(z)$,$V_A(z')$}:$y,z, z'\in$G,$zz'=yz = x$}
= isup{iinf {$V_B(z)$, $V_A(z')$}: $y, z, z'\in$ G, $x = zz'$}
= $V_{B\,o\,A}(x)$ by definition
Thus $V_{A\,o\,B}(x) = V_{B\,o\,A}(x)$ for each $x\in$ G.
Therefore A o B = B o A.

Similarly, suppose that B is an I-vague normal group of G. By the above we have B o A= A o B. Hence the theorem follows.

Remark: If G is an abelian group and A and B are I-vague groups of G, then A and B are I-vague normal groups of G.
Hence A o B = B o A.

Theorem 3.15: If I is infinitely meet distributive, then the product of I-vague sets of a group G is associative.

Proof: Let A = (t_A, f_A), B = (t_B, f_B) and C = (t_C, f_C) be I-vague sets of G. We prove that $V_{(A\,o\,B)oC}(x)$=$V_{A\,o\,(B\,o\,C)}(x)$for each $x\in$ G.
Let $x\in$ G. Then,
$V_{(A\,o\,B)oC}(x)$ = isup{iinf{$V_{AoB}(y)$, $V_C(z)$}: $y, z\in$G, $x = yz$}
=isup{iinf{isup{iinf{$V_A(u)$, $V_B(v)$}: $u, v\in$G, $y = uv$}, $V_C(z)$}: $y, z\in$G, $x = yz$}
=isup{iinf{[$\vee_{y=uv}(t_A(u) \wedge t_B(v))$, $\vee_{y=uv}((1-f_A(u) \wedge (1-f_B(v))]$, [$t_C(z)$, 1 - $f_C(z)$], $x = yz$}
=[$\vee_{x=yz}\{ \vee_{y=uv}(t_A(u) \wedge t_B(v)) \wedge t_C(z)\}$,$\vee_{x=yz}\{ \vee_{y=uv}\left((1-f_A(u)) \wedge (1-f_B(v)\right) \wedge (1-f_C(z))\}$]
=[$\vee_{x=up}\, t_A(u)\wedge\{\vee_{p=vz}(t_B(v) \wedge t_C(z))\}$, $\vee_{x=up}(1-f_A(u))\wedge\{\vee_{p=vz}((1-f_B(v)) \wedge (1- f_C(z))\}$]
= $V_{A\,o\,(B\,o\,C)}(x)$
Hence $V_{(A\,o\,B)oC}(x)$=$V_{A\,o\,(B\,o\,C)}(x)$for each $x\in$ G.
Therefore (A o B) o C = A o (B o C).

Theorem 3.16: Let I be infinitely meet distributive. Let A and B be I-vague groups of a group G. Then A o B = B o A if A o B is an I-vague group of G.

Proof: Let A and B be I-vague groups of G. Suppose that A o B = B o A,
To show that A o B is an I-vague group of G, we check
i) A o B =(A o B) o(A o B)
ii) $V_{A\,o\,B}(x)= V_{B\,o\,A}(x^{-1})$for each $x\in$ G.
i) Since A and B be I-vague groups of G,A o A = A and B o B = B.
A o B = (A o A) o (B o B)
= A o [A o (B o B)]
= A o [(A o B) o B)]
= A o [(B o A) o B]
= A o [B o (A o B)]
= (A o B) o (A o B)
This completes the proof of (i)

To prove (ii) let $x\in$ G. Then
$V_{A\,o\,B}(x)$ = isup{iinf{$V_A(y)$, $V_B(z)$}: $y, z\in$G, $x = yz$}
= isup{iinf{$V_A(y)$,$V_B(z)$}:y^{-1},$z^{-1}\in$G, $x^{-1} = z^{-1}y^{-1}$ }
= isup{iinf{$V_B(z)$,$V_A(y)$}: y^{-1}, $z^{-1}\in$G,$x^{-1} = z^{-1}y^{-1}$ }
= isup{iinf{$V_B(z^{-1})$,$V_A(y^{-1})$}:y^{-1}, $z^{-1}\in$G,$x^{-1}= z^{-1}y^{-1}$ }
= $V_{B\,o\,A}(x^{-1})$by definition.
Therefore $V_{A\,o\,B}(x)= V_{B\,o\,A}(x^{-1})$ for each $x\in$ G.
Since A o B = B o A by our assumption,
$V_{A\,o\,B}(x^{-1})= V_{B\,o\,A}(x^{-1})$ for each $x\in$ G.
It follows that $V_{A\,o\,B}(x)$=$V_{A\,o\,B}(x^{-1})$for each $x\in$ G.
By theorem 3.10, A o B is an I-vague group of G.

Conversely, suppose that A o B is an I-vague group of G.
We prove that $V_{A\,o\,B}(x)$=$V_{B\,o\,A}(x)$for each $x\in$ G.
$V_{A\,o\,B}(x) = V_{A\,o\,B}(x^{-1})$
= isup{iinf{$V_A(y)$, $V_B(z)$}: $y, z\in$G, $x^{-1}= yz$}
= isup{iinf{$V_A(y^{-1})$, $V_B(z^{-1})$}: $y, z\in$G, $x = z^{-1}y^{-1}$}
= isup{iinf{$V_B(z^{-1})$, $V_A(y^{-1})$}:y^{-1}, $z^{-1}\in$G, x=$z^{-1}y^{-1}$}
= $V_{B\,o\,A}(x)$
Hence $V_{A\,o\,B}(x)$=$V_{B\,o\,A}(x)$for each $x\in$ G.
Therefore A o B = B o A.

Corollary 3.17: Let I be infinitely meet distributive. Let A and B be I-vague groups of a group G. If either A or B is an I-vague normal group of G, then A o B is an I-vague group of G.

Proof: If either A or B is an I-vague normal group of G, then A o B =B o A by theorem 3.14. By theorem 3.16, A o B is an I-vague group of G.

Theorem 3.18: Let I be infinitely meet distributive. If A and B be I-vague normal groups of a group G, then A o B is an I-vague normal group of G.

Proof: Let A and B be I-vague normal groups of G. By corollary 3.17,A o B is an I-vague group of G. Now we show that $V_{A\,o\,B}(x)=V_{A\,o\,B}(y^{-1}xy)$for all $x, y\in$ G.

$V_{A\,o\,B}(x)$= isup{iinf{$V_A(p)$, $V_B(q)$}: $p, q\in$G, $x = pq$}

= isup{iinf{$V_A(y^{-1}py)$, $V_B(y^{-1}qy)$}: $p, q, y \in$G, $x = pq$}

= isup{iinf {$V_A(y^{-1}py)$, $V_B(y^{-1}qy)$}: $p, q, y \in$G, $y^{-1}xy = y^{-1}py\, y^{-1}qy$}

= $V_{A\,o\,B}(y^{-1}xy)$

Thus $V_{A\,o\,B}(x)=V_{A\,o\,B}(y^{-1}xy)$for all $x, y\in$ G.

Hence A o B is an I-vague normal group of G.

Corollary 3.19: Let $N_I(G)$ be the set of all I-vague normal groups of a group G. If I is infinitely meet distributive, then ($N_I(G)$, o) is a semi lattice.

Proof: Let A, B and C$\in N_I(G)$. Then A o B $\in N_I(G)$ by theorem 3.18. By lemma 3.8, A o A = A. By theorem 3.14, A o B= B o A.

Moreover, A o (B o C) =(A o B) o C by theorem 3.15. Therefore ($N_I(G)$, o) is a semi lattice.

Theorem 3.20: Let I be infinitely meet distributive and A, B and C be I-vague groups of a group G. If A$\subseteq$ C, then

A o (B $\cap$ C) = C $\cap$ (A o B).

Proof: Let A, B and C be I-vague groups of a group G. Suppose that A $\subseteq$ C.

We prove that A o (B $\cap$ C) =C $\cap$ (A o B).

Step 1: First we prove that

A o (B $\cap$ C) $\subseteq$C $\cap$ (A o B).

Let $x\in$G. Then

$V_{C\cap(A o B)}(x)$= iinf{$V_C(x)$, $V_{AoB}(x)$}

= iinf{$V_C(x)$, isup{iinf{$V_A(y)$, $V_B(z)$}: $y, z\in$G, x=yz}}

=iinf{$V_C(yz)$, isup{iinf{$V_A(y)$, $V_B(z)$}}: $y, z\in$G, x=yz}}

= isup{iinf{$V_C(yz)$, iinf{$V_A(y)$, $V_B(z)$}}: $y, z\in$G, x=yz}

≥isup{iinf{iinf{$V_C(y)$,$V_C(z)$},iinf{$V_A(y)$, $V_B(z)$}}: $y, z\in$G, x=yz}

=isup{iinf{iinf{$V_C(y)$,$V_A(y)$},inf{$V_C(z)$, $V_B(z)$}}:$y, z\in$G, x=yz}

=isup{iinf{$V_A(y)$, iinf{$V_C(z)$, $V_B(z)$}}: $y, z\in$G, x=yz} since A $\subseteq$ C.

= isup{iinf{$V_A(y)$, $V_{B\cap C}(z)$}: $y, z\in$ G, x=yz}

= $V_{A\,o\,(B\cap C)}(x)$ by definition

Hence $V_{A\,o\,(B\cap C)}(x) \le V_{C\cap(A o B)}(x)$ for all $x\in$G.

Therefore A o (B$\cap$ C) $\subseteq$C $\cap$ (A o B).

Step 2. Now we prove that

C $\cap$ (A o B)$\subseteq$A o (B$\cap$ C)

Let $x\in$G. Then

$V_{A\,o\,(B\cap C)}(x)$ = isup{iinf{$V_A(y)$, $V_{B\cap C}(z)$}: $y, z\in$G, x=yz}

=isup{iinf{$V_A(y)$, iinf{$V_B(z)$, $V_C(z)$}}: $y, z\in$G, x=yz }

=isup{iinf{$V_A(y)$,iinf{$V_B(z)$,$V_C(y^{-1}x)$}}: y^{-1}, $z\in$G, $z = y^{-1}x$}

≥isup{iinf{$V_A(y)$, iinf{$V_B(z)$, iinf{$V_C(y^{-1})$, $V_C(x)$}}}: y^{-1}, $z\in$G, z= y^{-1}x}

=isup{iinf{iinf{$V_A(y)$,$V_C(y^{-1})$}, iinf{$V_B(z)$,$V_C(x)$}}: $y, z\in$G, x= yz}

=isup{iinf{iinf{$V_A(y)$},$V_C(y)$}, iinf{$V_B(z)$,$V_C(x)$}}: $y, z\in$G, x= yz}

= isup{iinf{$V_A(y)$, iinf{$V_B(z)$,$V_C(x)$}}: $y, z\in$G, x= yz} since A $\subseteq$ C

=isup{iinf{$V_C(x)$, iinf{$V_A(y)$,$V_B(z)$}}: y,$z\in$G, x= yz}

=iinf{$V_C(x)$, isup{iinf{$V_A(y)$,$V_B(z)$}: y, $z\in$G, x= yz}}

= iinf{$V_C(x)$, $V_{AoB}(x)$}

=$V_{C\cap(A o B)}(x)$

Thus $V_{C\cap(A o B)}(x) \le V_{A\,o\,(B\cap C)}(x)$ for each$x\in$G.

Hence C $\cap$ (A o B)$\subseteq$A o (B$\cap$ C).

From step (1) and step (2), we have

A o (B $\cap$ C)= C $\cap$ (A o B).

Hence the theorem follows.

Notation: Let $N_{Ie}(G)$ denotes the set of all I-vague normal groups of a group G whose I-vague values at e are equal. Then we have the following corollary.

Corollary 3.21: If I is infinitely meet distributive, then ($N_{Ie}(G)$, $\subseteq$) is a modular lattice.

Proof: We prove that ($N_{Ie}(G)$, $\subseteq$) is a modular lattice.

Let A, B, C$\in N_{Ie}(G)$.

First we show that A$\cap$B, A o B$\in N_{Ie}(G)$.

A$\cap$B is an I-vague normal group of G by theorem 2.23. Moreover, A o B is an I-vague normal group of G by theorem 3.18.

Since A, B$\in N_{Ie}(G)$, $V_A(e)$ = $V_B(e)$.

$V_{A\cap B}(e)$ = iinf{$V_A(e)$,$V_B(e)$}= $V_A(e)$ = $V_B(e)$. Hence A$\cap$B $\in N_{Ie}(G)$.

$V_{A\,o\,B}(e)$ = isup{iinf{$V_A(x)$,$V_B(x^{-1})$: $x\in$G}

= iinf{$V_A(e)$,$V_B(e)$}

= $V_A(e)$

Hence A o B $\in N_{Ie}(G)$.

Consider ($N_{Ie}(G)$, $\subseteq$). It is a lattice where

A$\vee$ B = A o B by theorem 3.12 and

A $\wedge$ B = A $\cap$ B.

$A \subseteq C$ implies $A\ o(B \cap C) = C \cap (A\ o\ B)$ by theorem 3.20.

Hence $(N_{Ie}(G), \subseteq)$ is a modular lattice.

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