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TWO JACKKNIFE ESTIMATORS FOR THE SIGNAL AND COMPARISON OF THEIR PERFORMANCE BY A MONTE CARLO SIMULATION

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ABSTRACT: The paper introduces two jackknife estimators of the signal. The mean square errors of the proposed estimators and two other estimators of the signal are studied by simulation. The results of the study show that in designs having weak to mild collinearities both jackknife estimators are more efficient than the naive estimator as well as non-negative minimum biased estimator. The non-negative minimum biased estimator performs better under strong collinearity.

Key words/phrases: Jackknife estimators, mean square error, non negative minimum biased estimator, naive estimator, pseudo-values

INTRODUCTION

Consider the linear regression model $M\{y,X\beta,\sigma^2 I_n\}$, where y is a random vector of length n, β is a fixed but unknown p-vector of regression coefficients and $\sigma^2 > 0$ is the unknown variance of error terms.

Assume that the $n \times p$ non-stochastic matrix X has full column rank. The standard linear unbiased estimator of the vector of regression coefficients is the ordinary least squares estimator (OLSE) given by $b = (X'X)^{-1}X'y$.

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A resampling method called jackknifing can be used to provide a pooled estimator for ß. The *delete*-1 jackknife estimator b* is simply the average of $b_{-i} = (\mathbf{X}'_{-i}\mathbf{X}_{-i})^{-1}\mathbf{X}_{-i}'y_{-i}$, $i = 1, 2, \ldots, n$, where b_{-i} is the OLSE estimator obtained after deleting the ith row in the $n \times (p+1)$ matrix $(y,\mathbf{X})$. Similarly, we can think of a *delete*-k, $1 < k < n$, estimator of ß. One can imagine the difficulties involved in the delete-more-than one row jackknifing. The present paper introduces two estimators for the non-linear function of the regression coefficients commonly referred to as the signal $S = ß'ß$. This parametric function appears in the statistical literature on econometrics and regression analysis - quite often in ridge-type estimation (see among others Trenkler, D. (1986) and Trenkler, G. (1981)). The derivation of the two estimators utilises pseudo-values that are functions of the delete-1 jackknife estimator b_{-i}.

ESTIMATORS OF THE SIGNAL

The most plausible and meaningful estimators of the signal belong to the class

$$F = \{\hat{S} \mid \hat{S} = b'\mathbf{W}b, \ \mathbf{W} \text{ is a weight matrix}\}.$$

A member of F is the *naive* estimator b'b (with $\mathbf{W} = \mathbf{I}_p$). This estimator is biased, and under multicollinearity overestimates S distinctly. An unbiased estimator of S is $b'b - s^2 \Sigma_{j=1}^{p} l_j^{-1}$ where s^2 is the unbiased estimator of the noise σ^2 and l_j, $j = 1, 2, \ldots, p$ are the eigenvalues of $\mathbf{X}'\mathbf{X}$.

An undesirable property of this estimator is that it is likely to assume negative values in the presence of multicollinearity (McDonald and Galarneau, 1975; Brook and Moore, 1980; Trenkler, 1981). Gnot *et al.* (1995) derived a non-negative minimum biased estimator for the parameter function $ß'\mathbf{H}ß + h\sigma^2$ where $\mathbf{H}$ is a non-negative definite matrix and the scalar $h \geq 0$. The estimator has the form $b'\mathbf{C}_H b$ with $\mathbf{C}_H = \mathbf{H} - \lambda(\mathbf{X}'\mathbf{X})^+$. The constant $\lambda = tr[\mathbf{C}_H(\mathbf{X}'\mathbf{X})^+] - h$ and $(\mathbf{X}'\mathbf{X})^+$ is the Moore-Penrose inverse of $\mathbf{X}'\mathbf{X}$.

Applying the above general result of Gnot *et al.* (1995) to the full rank model with $\mathbf{H} = \mathbf{I}_p$ and $h = 0$ we obtain the non-negative minimum biased estimator for the signal, namely

$$b'[I_p - \lambda^*(X'X)^{-1}]b,$$

where the constant

$$\lambda^* = \frac{tr(X'X)^{-1}}{1 + tr(X'X)^{-2}}.$$

This estimator belongs to the class F with $\mathbf{W} = \mathbf{I}_p - \lambda^*(\mathbf{X}'\mathbf{X})^{-1}$.

The present paper has two purposes. First, it introduces two jackknife estimators of the signal that are functions of the *delete*-1 estimator of ß. Secondly, it compares the performance of the two jackknife estimators that we will introduce below, the minimum biased estimator, the last estimator given above and that of the naive estimator on the basis of a simulation study.

The idea of obtaining jackknife estimators is based on what are referred to as *pseudo-values*. These are the essential and basic elements used in the derivation of jackknife estimators of a parameter function (Miller 1974a; 1974b; Hinkley, 1977).

Since the discussion here pertains to a multiple linear regression model we distinguish between an unbalanced and a balanced model (Belsley *et al.*, 1980; Belsley, 1991) before introducing the *pseudo-values*.

First of all it is important to note that the relationship between b and b_{-i} can be given by

$$b_{-i} = b - (1 - h_i)^{-1} (\mathbf{X}'\mathbf{X})^{-1} x_i e_i, \; i \doteq 1,2, \ldots, n$$

where $h_i = x_i'(\mathbf{X}'\mathbf{X})^{-1} x_i$ is the ith diagonal element of the least squares projection (hat) matrix $\mathbf{H} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$, with $h_i \in [0,1]$, $i = 1,2, \ldots, \underline{n}$, and,

e_i is the *i*th component of there residual vector $\boldsymbol{e} = \boldsymbol{y} - \hat{\boldsymbol{y}}$. Since by assumption the matrix **X** is of full column rank $\sum_{i=1}^{n} h_i = p$. Belseley *et al.* (1980) say the following about the size of the diagonal elements of the projection matrix **H**: "The average size of a diagonal element h_i, ..., is p/n. ... if we were designing an experiment, it would be desirable to use data that were roughly equally influential, that is, each observation having an h_i near to the average p/n ...". Since, in reality, the data matrices available to us are not generated from designed experiments we make a distinction between what we call a balanced and an unbalanced model. A model is called *perfectly balanced* if $h_i = p/n$. In practice those cases with h_i close to p/n are considered as *balanced.* In situations where the difference among the h_i values is large the underlying model is said to be *unbalanced.*

We distinguish among three magnitudes of h_i. Firstly, $h_i = 0$ means that the OLSE cannot be used since the matrix $\mathbf{X}_{-i}$ will be rank deficient. Second, the other extreme situation where $h_i = 1$ implies that $\hat{y}_i = y_i$, that is $e_i = 0$, i=1,2,...,n. This means that one parameter is determined completely by y_i or, in effect, dedicated to one data point. Third, the special case $h_i > 2p/n$ plays an important role in regression diagnostics. The ith observation is a *leverage point* when $h_i > 2p/n$. With these words we complete the above brief discussion on balanced and unbalanced models.

We now proceed with the two *pseudo-values* - one for balanced and one for the unbalanced model. For the situation with a balanced design model the *pseudo-values*

$$P_i = nb'b - (n - 1)b'_{-i}b_{-i}, \quad i=1,2, \ldots,n$$

can be used so that we get the estimator

$$\hat{S}_1 = nb'b - \left(\frac{n-1}{n}\right)\sum_{i=1}^{n} b_{-i}b_{-i}.$$

For the unbalanced design model the *pseudo-values* are

$$Q_i = b'b - n(1 - h_i)(b'_{-i} - b_{-i}-b'b), \quad i=1,2, \ldots,n,$$

and the resulting signal estimator is

$$\hat{S}_2 = b'b - \sum_{i=1}^{n} (1-h_i)(b_{-i}b_{-i} - b'b)$$

Finally, for the sake of consistency we designate the naive estimator and the non-negative minimum biased estimator by $\hat{S}_3, \hat{S}_4$, respectively.

THE SIMULATION STUDY

The models

We follow procedures that are similar to those introduced in McDonald and Galarneau (1975), Wichern and Churchill (1978) and Gibbons (1981) to construct the regression matrices and error vectors in the two models that will be considered.

The columns of the 50×5 matrix **M** and those of the 100×5 matrix **N** are standard normal distributed *pseudo-random* numbers, that is, u_{i1}, u_{i2}, u_{i3}, u_{i4}, u_{i5}, $u_{i6} \sim N(0,1)$. These are generated according to an algorithm by Box and Muller (1958), and held fixed throughout the simulation. Thus, every matrix **X** that will be used in the simulation can be obtained by controlling the correlations among the columns of **M** and **N** as follows:

$$(1-\varrho^2)^{1/2}u_{ij} + \varrho u_{i6}, \quad i = 1,2,\ldots,n; \quad j = 1,2,3$$
$$(1-\varrho^2_*)^{1/2}u_{ij} + \varrho_* u_{i6}, \quad i = 1,2,\ldots,n; \quad j = 4,5$$

where $\varrho, \varrho_* \in (-1,1)$, and $n = 50$ and $n = 100$.

As a result ϱ^2 gives the magnitude of correlation among the first three regressors while ϱ^2_* is the magnitude of correlation between the fourth and fifth regressors. On the other hand, the product $\varrho\varrho_*$ shows the size as well as the type of correlations between any one of the first three and any of the last two exogenous variables.

The study considered a total of 121 configurations (for each $n = 50$ and $n = 100$) of (ϱ, ϱ_*) that represent different levels of collinearity. These range from ($\varrho = 0$, $\varrho_* = 0$) (the ideal orthogonal design) to ($\varrho = 0.9995$, $\varrho_* = 0.9995$) (near-perfect correlation).

In the simulation the standardised (centred and scaled) regression matrices $\mathbf{Z}_{50}$ and $\mathbf{Z}_{100}$ (corresponding to the two matrices X) have been used.

We used the orthonormal eigenvectors P_{max} and P_{min} that correspond to the largest and smallest eigenvalues, respectively, of $\mathbf{Z'Z}$ as the vectors of regression coefficients (see McDonald and Galarneau (1975)). The standard deviations $\sigma = 0.10, 0.15, 0.25, 0.50, 1.00, 2.00, 4.44$ were used to represent a wide spectrum of variability of the error terms.

Without making any reference to dimensions the two models used in the simulation can be given as: $M\{y, Zp_{max}, \sigma^2 I_n\}$ and $M\{y, Zp_{min}, \sigma^2 I_n\}$.

INTERPRETATION OF THE RESULTS AND RÉSUME

The results

The empirical mean square error of an estimator $\hat{S}$ for a total of m simulation runs is computed as $m^{-1}\sum_{r=1}^{m}(\hat{S}_{(r)} - \beta'\beta)'(\hat{S}_{(r)} - \beta'\beta)$, where $\hat{S}_{(r)}$ is the estimated squared norm of the estimator in the rth run. Note that here $\beta'\beta$ is identically equal to 1, because in the simulation models $\beta'\beta = p_{max}'p_{max} = p_{min}'p_{min} = 1$.

For selected pairs of (ϱ, ϱ_*) (10 pairs for $n = 50$ and another 10 pairs for $n = 100$) and all standard deviations given in column two the empirical mean square errors of the estimators based on 1000 runs are given in the Appendix. The last four columns in Tables 1 and 3 are the simulated mean square errors of $S1_{max}$, $S2_{max}$, $S3_{max}$, and $S4_{max}$ for $n = 50$ and 100, respectively. Similarly, in Tables 2 and 4 we find the mean square erors of $S1_{min}$, $S2_{min}$, $S3_{min}$ and $S4_{min}$.

RÉSUMÉ

General observations

(a) The mean square errors associated with p_{max} for each design and each estimator were found to be smaller than the corresponding errors associated with p_{min}. This is a direct theoretical consequence that is attributed to the fact that the maximum data concentration (in the data space) is along the eigenvector that corresponds to the largest eigenvalue, thereby providing the best estimate for ß. This, as a result, can contribute to an improvement in the estimation of the signal. On the other hand, p_{max} is expected to have the opposite effect on the estimation of ß. Similar observations were noted in the simulation studies by McDonald and Galarneau (1975) and Wichern and Churchill (1978).

(b) The relatively larger variances have contributed to larger mean square errors in all designs. This phenomenon becomes more severe the greater the level of collinearity, and it is more evident for the values $\sigma = 2$ and $\sigma = 4$.

(c) The above two general comments hold for both designs with $n = 50$ and $n = 100$. In fact, there is not much we have learned by taking the design with $n = 100$ except that in the designs with this dimension an increase in efficiency is observed for all estimators.

Specific observations

(a) The mean square errors of all four estimators for the orthogonal design, that is $(\varrho, \varrho_*) = (0, 0)$, did not differ very much, and, in fact, the values are very small. This means all four estimators perform equally well. For the sake of simplicity the naive estimator can be used in applications.

(b) For the cases $(\varrho, \varrho_*) = (0, 0.50)$, $(0.50, 0)$, $(0.50, 0.50)$ and $(0.75, 0.75)$ the jackknife estimator $\hat{S}_2$ is the best, $\hat{S}_1$ is the second best whereas the remaining two have almost the same level of efficiency.

(c) The non-negative minimum biased estimator $\hat{S}_4$ by far excels all three for the values $(\varrho, \varrho_*) = (0.50, 0.995), (0.995, 0), (0.995, 0.50)$ and $(0.995, 0.995)$. This has the direct implication that $\hat{S}_4$ is an appropriate estimator for the signal when multicollinearity prevails.

(d) The above three conclusions apply to both designs with $n=50$ and $n=100$.

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APPENDIX: THE SIMULATION RESULTS.

Table 1. Mean square error of the four estimators ($n=50$, $p=5$). ß = eigenvector corresponding to the largest eigenvalue of X'X.

ρ	ρ^*	σ	$s1_{max}$	$s2_{max}$	$s3_{max}$	$s4_{max}$
0	0	0.1	0.0005951	0.0005949	0.0005972	0.0005925
0	0	0.15	0.0013373	0.0013262	0.001336	0.0013229
0	0	0.25	0.0036923	0.0036856	0.003684	0.0036607
0	0	0.5	0.0158004	0.0155919	0.0161037	0.015975
0	0	1	0.0655526	0.0636642	0.0751322	0.0744852
0	0	2	0.3443206	0.330329	0.524735	0.5208839
0	0	4	2.8313237	2.6816868	5.7872476	5.7505785
0	0.5	0.1	0.0005551	0.0005509	0.0005571	0.000548
0	0.5	0.15	0.0013177	0.001307	0.0013144	0.001303
0	0.5	0.25	0.0035925	0.0035571	0.003577	0.0035492
0	0.5	0.5	0.0132421	0.0132306	0.0139419	0.0138114
0	0.5	1	0.0680971	0.0661088	0.0781053	0.07744
0	0.5	2	0.3405564	0.3304584	0.5357989	0.5316041
0	0.5	4	2.904014	2.7148329	5.8203232	5.7810902
0	0.995	0.1	0.0018596	0.0018307	0.0023346	0.0004284
0	0.995	0.15	0.007229	0.0069252	0.0096652	0.001006
0	0.995	0.25	0.055299	0.0536157	0.0771111	0.0037485
0	0.995	0.5	0.784636	0.7278162	1.063564	0.0287995
0	0.995	1	14.20097	13.464537	18.666295	0.4276977
0	0.995	2	215.19594	204.16224	275.2817	5.9278741
0	0.995	4	3062.8618	2905.0703	4288.9543	99.675889

Table 1. (Contd.)

ρ	ρ^*	σ	$s1_{max}$	$s2_{max}$	$s3_{max}$	$s4_{max}$
0.5	0	0.1	0.0005273	0.0005292	0.0005319	0.0005277
0.5	0	0.15	0.0011592	0.0011486	0.0011556	0.0011469
0.5	0	0.25	0.0033601	0.0033309	0.0033585	0.0033348
0.5	0	0.5	0.013245	0.0130976	0.014151	0.014002
0.5	0	1	0.0594434	0.0590038	0.0735665	0.0728757
0.5	0	2	0.3607015	0.3520864	0.566392	0.5618003
0.5	0	4	3.4470764	3.2171461	6.4545866	6.4076386
0.5	0.5	0.1	0.0005091	0.0005033	0.0005044	0.0005026
0.5	0.5	0.15	0.0011346	0.0011229	0.0011319	0.0011213
0.5	0.5	0.25	0.0033774	0.003328	0.0033588	0.003332
0.5	0.5	0.5	0.0135275	0.0133725	0.0143926	0.0142484
0.5	0.5	1	0.059318	0.0580256	0.0705052	0.0698406
0.5	0.5	2	0.3502921	0.3347974	0.5522625	0.5475827
0.5	0.5	4	3.1181091	2.8322366	6.1168304	6.0689048
0.5	0.995	0.1	0.0020046	0.0018569	0.0024559	0.0004025
0.5	0.995	0.15	0.0069592	0.0069529	0.0101598	0.0010026
0.5	0.995	0.25	0.0561141	0.0539865	0.0761452	0.0038636
0.5	0.995	0.5	0.812146	0.794128	1.1720685	0.0301272
0.5	0.995	1	12.99633	12.245341	17.816729	0.4166275
0.5	0.995	2	204.77372	191.96362	258.65156	6.2269041
0.5	0.995	4	3758.6147	3613.4881	5055.1129	93.09735
0.75	0.75	0.1	0.0003689	0.0003677	0.0003716	0.0003671
0.75	0.75	0.15	0.0008415	0.0008308	0.0008473	0.0008338
0.75	0.75	0.25	0.0023049	0.0022839	0.00244	0.00239
0.75	0.75	0.5	0.01123	0.0110366	0.01325	0.0129898
0.75	0.75	1	0.0602501	0.0590011	0.0897262	0.0878627
0.75	0.75	2	0.5188961	0.4852643	0.9536507	0.9331916
0.75	0.75	4	6.1453193	5.7114345	13.733163	13.453969

Table 1. (Contd.)

ρ	ρ^*	σ	$s1_{max}$	$s2_{max}$	$s3_{max}$	$s4_{max}$
0.995	0	0.1	0.0032769	0.003121	0.0055078	0.0003366
0.995	0	0.15	0.0156547	0.0148185	0.0252222	0.000954
0.995	0	0.25	0.1026718	0.0973627	0.1763105	0.0048011
0.995	0	0.5	1.8363284	1.7423609	3.1626165	0.0680357
0.995	0	1	23.264072	21.246541	41.463447	0.7123986
0.995	0	2	400.78824	360.4004	642.12702	12.242537
0.995	0	4	7627.4309	7169.8206	13264.66	273.51804
0.995	0.5	0.1	0.0034446	0.0032829	0.0055435	0.0003336
0.995	0.5	0.15	0.0145148	0.013183	0.0222707	0.0008751
0.995	0.5	0.25	0.1043739	0.0964093	0.1719625	0.0046552
0.995	0.5	0.5	1.8921878	1.7406322	2.9811838	0.0653818
0.995	0.5	1	30.460626	29.577837	52.773914	1.0317636
0.995	0.5	2	451.27547	422.94167	802.16697	15.457104
0.995	0.5	4	8716.1677	8325.9735	14453.049	231.738
0.995	0.995	0.1	0.005257	0.0049587	0.0101595	0.0005181
0.995	0.995	0.15	0.0213817	0.0199687	0.0433674	0.0017555
0.995	0.995	0.25	0.1597266	0.1444727	0.3101745	0.0097295
0.995	0.995	0.5	2.4915976	2.2591745	4.9092783	0.1346179
0.995	0.995	1	44.940461	42.948424	94.236206	2.3874944
0.995	0.995	2	727.04489	680.51839	1434.0659	39.139709
0.995	0.995	4	11110.098	10271.946	22270.696	616.93695

Table 2. ß = eigenvector corresponding to the smallest eigenvalue of X'X.

ρ	ρ^*	σ	$s1_{min}$	$s2_{min}$	$s3_{min}$	$s4_{min}$
0	0	0.1	0.0011666	0.0011612	0.001167	0.0011522
0	0	0.15	0.0030563	0.0030149	0.0030142	0.0029858
0	0	0.25	0.0081663	0.008054	0.0080632	0.0079879
0	0	0.5	0.034983	0.034555	0.0348614	0.0344716
0	0	1	0.1420395	0.1407119	0.14867	0.1468754
0	0	2	0.7038554	0.695771	0.9466886	0.9361283
0	0	4	4.2911568	4.1021324	7.1940104	7.1345649

Table 2. (Contd.)

ρ	ρ^*	σ	$s1_{min}$	$s2_{min}$	$s3_{min}$	$s4_{min}$
0	0.5	0.1	0.001395	0.0013735	0.0013723	0.0013746
0	0.5	0.15	0.0031958	0.0031811	0.0031952	0.0031486
0	0.5	0.25	0.0092859	0.0092154	0.009284	0.0091372
0	0.5	0.5	0.0381313	0.038182	0.0390315	0.0384589
0	0.5	1	0.1478597	0.1458788	0.1601278	0.1578042
0	0.5	2	0.7568625	0.7414909	0.9296581	0.9186013
0	0.5	4	4.4668186	4.3375135	8.0625395	7.9909233
0	0.995	0.1	0.0957774	0.0950542	0.0961697	0.7189328
0	0.995	0.15	0.2216588	0.2201241	0.2216531	0.7109403
0	0.995	0.25	0.6493991	0.6482023	0.6758136	0.6916736
0	0.995	0.5	3.1133375	2.9937941	3.2164167	0.651556
0	0.995	1	21.401605	20.862052	26.379483	0.7053705
0	0.995	2	231.62019	215.09525	284.2678	4.4135939
0	0.995	4	3336.5428	3151.3152	4538.7018	92.832164
0.5	0	0.1	0.0015768	0.0015626	0.0015652	0.0015552
0.5	0	0.15	0.003597	0.0035739	0.0035696	0.0035518
0.5	0	0.25	0.0090403	0.0090915	0.0091904	0.0090422
0.5	0	0.5	0.0408503	0.040627	0.0410764	0.0404695
0.5	0	1	0.1673498	0.1676064	0.1821604	0.179313
0.5	0	2	0.8110255	0.8072654	1.0623998	1.0480314
0.5	0	4	4.6434062	4.4119477	7.6732257	7.5946167
0.5	0.5	0.1	0.0017513	0.0017419	0.0017411	0.0017463
0.5	0.5	0.15	0.0036266	0.0036129	0.0036183	0.0035749
0.5	0.5	0.25	0.0098234	0.0097505	0.0099316	0.0097041
0.5	0.5	0.5	0.0402221	0.039807	0.0403923	0.0397808
0.5	0.5	1	0.1671882	0.1650094	0.1757774	0.1728451
0.5	0.5	2	0.7670165	0.7487132	1.0024206	0.9877824
0.5	0.5	4	4.9585369	4.7393523	8.400766	8.3108596

Table 2. (Contd.)

ρ	ρ^*	σ	$s1_{min}$	$s2_{min}$	$s3_{min}$	$s4_{min}$
0.5	0.995	0.1	0.1006273	0.098978	0.1000767	0.7174963
0.5	0.995	0.15	0.2143524	0.2146863	0.2195959	0.714707
0.5	0.995	0.25	0.6610428	0.6550091	0.6835074	0.6944202
0.5	0.995	0.5	2.9318095	2.9210736	3.2404986	0.6476669
0.5	0.995	1	19.636304	18.815242	24.009623	0.6439973
0.5	0.995	2	236.47053	224.15823	292.03575	5.1546793
0.5	0.995	4	3884.2955	3730.1735	5213.3817	85.369388
0.75	0.75	0.1	0.0027836	0.0027505	0.0027513	0.0028141
0.75	0.75	0.15	0.00634	0.0062973	0.0063095	0.0062124
0.75	0.75	0.25	0.0184754	0.0184142	0.0186152	0.0178861
0.75	0.75	0.5	0.0720421	0.0709611	0.0721258	0.0692533
0.75	0.75	1	0.2954402	0.2927773	0.3154818	0.3028207
0.75	0.75	2	1.3540029	1.2957305	1.7576416	1.6966034
0.75	0.75	4	10.377214	9.8768821	18.019978	17.558162
0.995	0	0.1	0.1194926	0.1178983	0.11902	1.0831876
0.995	0	0.15	0.2695932	0.2675237	0.2788236	1.0697394
0.995	0	0.25	0.7635758	0.7479767	0.8278451	1.0333475
0.995	0	0.5	4.681787	4.5037392	5.8623182	0.9746813
0.995	0	1	35.907499	33.9268	54.067818	1.1762504
0.995	0	2	457.98225	417.99694	685.38894	10.454516
0.995	0	4	7753.8767	7314.23	13463.027	259.9022
0.995	0.5	0.1	0.1210953	0.1202996	0.1222868	1.0788156
0.995	0.5	0.15	0.2573737	0.2525333	0.2630235	1.0663835
0.995	0.5	0.25	0.827111	0.8114609	0.8969542	1.0562812
0.995	0.5	0.5	4.6658222	4.4763499	5.8888108	0.9632771
0.995	0.5	1	41.117339	40.421932	63.806161	1.3184318
0.995	0.5	2	506.22322	474.86151	855.38115	13.321891
0.995	0.5	4	9160.7768	8787.9593	15042.136	226.23856

Table 2. (Contd.)

ρ	ρ^*	σ	$s1_{min}$	$s2_{min}$	$s3_{min}$	$s4_{min}$
0.995	0.995	0.1	0.1398241	0.1395114	0.1442225	1.3789261
0.995	0.995	0.15	0.2947891	0.2901354	0.3142993	1.3751769
0.995	0.995	0.25	0.9688015	0.9552943	1.1563851	1.3848132
0.995	0.995	0.5	5.2669839	4.9215484	7.2431231	1.3153428
0.995	0.995	1	56.463822	54.265695	104.11421	2.950884
0.995	0.995	2	760.28705	710.9024	1466.1229	37.446463
0.995	0.995	4	11404.362	10553.834	22689.514	619.94244

Table 3. Mean square error of the four estimators ($n=100$, $p=5$). ß = eigenvector corresponding to the largest eigenvalue of X'X.

ρ	ρ^*	σ	$s1_{max}$	$s2_{max}$	$s3_{max}$	$s4_{max}$
0	0	0.1	0.0003116	0.00031	0.0003098	0.0003095
0	0	0.15	0.0007078	0.0007062	0.0007064	0.0007053
0	0	0.25	0.0020919	0.0020833	0.0020857	0.0020821
0	0	0.5	0.0083996	0.0083718	0.0085535	0.0085352
0	0	1	0.0336669	0.0335884	0.036157	0.0360833
0	0	2	0.1471984	0.1462424	0.1868612	0.1865149
0	0	4	0.8916232	0.8898289	1.6251612	1.622722
0	0.5	0.1	0.0002933	0.0002928	0.0002935	0.0002928
0	0.5	0.15	0.0006275	0.0006262	0.0006285	0.000627
0	0.5	0.25	0.0019347	0.0019362	0.0019346	0.0019319
0	0.5	0.5	0.0073234	0.0072685	0.0073915	0.0073762
0	0.5	1	0.0294041	0.0293061	0.0322355	0.0321655
0	0.5	2	0.1479412	0.1460262	0.1806129	0.1802854
0	0.5	4	0.7843612	0.7737074	1.4115742	1.4093243
0	0.995	0.1	0.000496	0.000491	0.0006126	0.0002515
0	0.995	0.15	0.0020196	0.0020115	0.002765	0.000817
0	0.995	0.25	0.0096204	0.0092611	0.0122603	0.0033843
0	0.995	0.5	0.1568929	0.1513044	0.2249492	0.0491038
0	0.995	1	2.5816063	2.5313277	3.6181738	0.7521186
0	0.995	2	40.046241	39.655328	59.13492	10.167187
0	0.995	4	575.12936	568.93403	875.65324	169.05378

Table 3. (Contd.)

ρ	ρ^*	σ	$s1_{max}$	$s2_{max}$	$s3_{max}$	$s4_{max}$
0.5	0	0.1	0.0002443	0.0002432	0.0002438	0.0002431
0.5	0	0.15	0.000559	0.0005574	0.0005561	0.000556
0.5	0	0.25	0.0016409	0.0016376	0.0016529	0.001649
0.5	0	0.5	0.0067565	0.0067358	0.0069001	0.0068853
0.5	0	1	0.0277968	0.0275937	0.0300919	0.0300281
0.5	0	2	0.1183325	0.1180257	0.1749272	0.1745656
0.5	0	4	0.8109091	0.800014	1.5647088	1.5620979
0.5	0.5	0.1	0.0002503	0.00025	0.0002502	0.0002498
0.5	0.5	0.15	0.000593	0.0005919	0.0005903	0.0005899
0.5	0.5	0.25	0.0016764	0.0016736	0.001683	0.0016795
0.5	0.5	0.5	0.0065895	0.006562	0.0066743	0.006661
0.5	0.5	1	0.0282492	0.0281531	0.031937	0.0318638
0.5	0.5	2	0.1314285	0.1302055	0.1747083	0.1743583
0.5	0.5	4	0.8134078	0.8013099	1.6089836	1.6061224
0.5	0.995	0.1	0.0004392	0.000435	0.0005361	0.0002621
0.5	0.995	0.15	0.0016957	0.0016765	0.0022526	0.0007296
0.5	0.995	0.25	0.0125026	0.0124493	0.0179609	0.0040675
0.5	0.995	0.5	0.1447096	0.1425686	0.2093953	0.0451616
0.5	0.995	1	2.7539116	2.7714743	4.0900261	0.7716224
0.5	0.995	2	36.220775	35.734426	53.884945	9.7395533
0.5	0.995	4	683.24166	682.83471	1035.0894	190.03129
0.75	0.75	0.1	0.0001834	0.0001828	0.0001839	0.0001832
0.75	0.75	0.15	0.0004439	0.0004432	0.0004521	0.0004491
0.75	0.75	0.25	0.0012171	0.0012148	0.0012348	0.0012304
0.75	0.75	0.5	0.0050715	0.0050673	0.0055206	0.0054972
0.75	0.75	1	0.0229915	0.0228717	0.0302362	0.030102
0.75	0.75	2	0.136915	0.1352336	0.2477287	0.2466836
0.75	0.75	4	1.3913846	1.361986	3.1721309	3.1586364
0.995	0	0.1	0.0006554	0.0006463	0.0011161	0.0001867
0.995	0	0.15	0.0027961	0.002777	0.0051991	0.0006681
0.995	0	0.25	0.0191147	0.0190453	0.0373353	0.0036316
0.995	0	0.5	0.3499022	0.3453208	0.6344977	0.0519708
0.995	0	1	5.1469309	5.0801193	9.8329445	0.8178521
0.995	0	2	98.877668	97.916341	176.38302	14.1864
0.995	0	4	1221.9248	1186.3308	2275.3407	212.36753

Table 3. (Contd.)

ρ	ρ^*	σ	$s1_{max}$	$s2_{max}$	$s3_{max}$	$s4_{max}$
0.995	0.5	0.1	0.0006678	0.0006601	0.001135	0.0001936
0.995	0.5	0.15	0.0024328	0.0023745	0.0043416	0.0006449
0.995	0.5	0.25	0.0184105	0.0180857	0.0367929	0.0035025
0.995	0.5	0.5	0.3851806	0.3814368	0.7037358	0.0552604
0.995	0.5	1	5.1462372	5.0405912	9.9606466	0.9304642
0.995	0.5	2	93.258481	91.987979	177.63873	15.672058
0.995	0.5	4	1379.2308	1354.9317	2482.7344	206.03215
0.995	0.995	0.1	0.0010124	0.0010027	0.0021961	0.0002119
0.995	0.995	0.15	0.0047868	0.0047329	0.0099851	0.0007114
0.995	0.995	0.25	0.0340356	0.0330464	0.0726943	0.0042123
0.995	0.995	0.5	0.4654353	0.4547857	1.0518902	0.060207
0.995	0.995	1	7.3275429	7.1557008	17.192611	0.9702485
0.995	0.995	2	120.81034	118.51653	275.02533	14.550016
0.995	0.995	4	2136.4675	2093.7139	4880.463	265.86138

Table 4. ß = eigenvector corresponding to the smallest eigenvaluė of **X'X**.

ρ	ρ^*	σ	$s1_{max}$	$s2_{max}$	$s3_{max}$	$s4_{max}$
0	0	0.1	0.0005721	0.0005691	0.000568	0.0005688
0	0	0.15	0.0012572	0.0012518	0.0012532	0.00125
0	0	0.25	0.003724	0.003726	0.003763	0.0037478
0	0	0.5	0.0132271	0.0131916	0.0132821	0.0132469
0	0	1	0.0549721	0.0551513	0.0573879	0.0572355
0	0	2	0.250056	0.2503076	0.2973828	0.2966396
0	0	4	1.2790492	1.270756	1.9829137	1.9790274
0	0.5	0.1	0.0005903	0.0005905	0.0005919	0.0005898
0	0.5	0.15	0.0013573	0.0013558	0.0013603	0.0013548
0	0.5	0.25	0.0038881	0.0038812	0.003878	0.0038711
0	0.5	0.5	0.0148028	0.0147576	0.0150753	0.015019
0	0.5	1	0.0635518	0.0635428	0.0658288	0.0656333
0	0.5	2	0.2466685	0.2448355	0.285108	0.2843025
0	0.5	4	1.2036015	1.1818022	1.738526	1.7346695

Table 4. (Contd.)

ρ	ρ^*	σ	$s1_{max}$	$s2_{max}$	$s3_{max}$	$s4_{max}$
0	0.995	0.1	0.0447781	0.0447353	0.044897	0.3105047
0	0.995	0.15	0.1129714	0.113085	0.1149285	0.3075439
0	0.995	0.25	0.2418022	0.2392693	0.242482	0.3190745
0	0.995	0.5	1.3156552	1.2985903	1.3822644	0.4411246
0	0.995	1	6.5372949	6.4365895	7.2256261	1.2621842
0	0.995	2	55.775873	55.274644	74.393386	11.176298
0	0.995	4	646.4837	638.88518	937.91196	172.633
0.5	0	0.1	0.0006665	0.0006637	0.0006645	0.0006625
0.5	0	0.15	0.0015849	0.0015769	0.0015741	0.0015734
0.5	0	0.25	0.0043709	0.0043774	0.004406	0.0043897
0.5	0	0.5	0.0158112	0.0158361	0.0160911	0.0160325
0.5	0	1	0.0692377	0.0690164	0.0719758	0.0717361
0.5	0	2	0.2853214	0.282725	0.3193595	0.3184052
0.5	0	4	1.5046534	1.4892529	2.2975992	2.2921815
0.5	0.5	0.1	0.0007034	0.0007021	0.0007034	0.0006999
0.5	0.5	0.15	0.0014951	0.0014945	0.0014982	0.0014924
0.5	0.5	0.25	0.0042966	0.0043047	0.0043162	0.0043
0.5	0.5	0.5	0.0172491	0.017244	0.0176212	0.017547
0.5	0.5	1	0.0725407	0.0724231	0.0760075	0.0757414
0.5	0.5	2	0.3240323	0.3232663	0.3815024	0.3802614
0.5	0.5	4	1.5577563	1.5508626	2.3899816	2.3838444
0.5	0.995	0.1	0.0397629	0.039698	0.0400102	0.3027908
0.5	0.995	0.15	0.0972118	0.0971453	0.0981536	0.3099425
0.5	0.995	0.25	0.3004987	0.300192	0.3070047	0.3314634
0.5	0.995	0.5	1.1619922	1.1596795	1.2145881	0.4428499
0.5	0.995	1	6.8782973	6.9238525	8.3982001	1.3820838
0.5	0.995	2	52.521175	52.109698	70.223039	10.991993
0.5	0.995	4	780.12018	780.80219	1131.8988	197.40153

Table 4. (Contd.)

ρ	ρ^*	σ	$s1_{max}$	$s2_{max}$	$s3_{max}$	$s4_{max}$
0.75	0.75	0.1	0.0012466	0.0012395	0.0012406	0.0012344
0.75	0.75	0.15	0.0027248	0.002724	0.0027258	0.0027108
0.75	0.75	0.25	0.0076029	0.0075921	0.0075935	0.0075426
0.75	0.75	0.5	0.0308895	0.0308269	0.0311749	0.0309211
0.75	0.75	1	0.1242749	0.1241535	0.131479	0.1303513
0.75	0.75	2	0.5351333	0.5305204	0.6346755	0.6299473
0.75	0.75	4	3.0959713	3.0609717	4.758131	4.7292696
0.995	0	0.1	0.0484657	0.048482	0.0487489	0.6034714
0.995	0	0.15	0.1064465	0.106239	0.1093413	0.5841132
0.995	0	0.25	0.3151817	0.3159286	0.33313	0.5641042
0.995	0	0.5	1.43916	1.4354837	1.7417877	0.4691627
0.995	0	1	10.02463	9.9924344	14.712354	0.6474649
0.995	0	2	113.29629	112.2827	193.45421	11.773259
0.995	0	4	1236.4252	1198.124	2271.6412	197.42007
0.995	0.5	0.1	0.0479426	0.0476873	0.0481837	0.6007423
0.995	0.5	0.15	0.0990338	0.0990739	0.1008734	0.5953529
0.995	0.5	0.25	0.3250705	0.3241137	0.3398461	0.5641866
0.995	0.5	0.5	1.9190939	1.9117052	2.3643787	0.4533881
0.995	0.5	1	9.8319071	9.6048897	13.823706	0.7358527
0.995	0.5	2	120.37393	119.09647	206.22607	13.532898
0.995	0.5	4	1413.4224	1390.2734	2503.0798	192.65816
0.995	0.995	0.1	0.0601401	0.0602035	0.0611059	0.8617689
0.995	0.995	0.15	0.1321415	0.1317119	0.1395461	0.8480429
0.995	0.995	0.25	0.3860851	0.3826165	0.4224454	0.7920356
0.995	0.995	0.5	1.9376664	1.9178998	2.4949652	0.6109328
0.995	0.995	1	11.765444	11.575506	21.310945	0.5529718
0.995	0.995	2	141.92583	139.88618	296.32968	10.35447
0.995	0.995	4	2290.4973	2249.9995	5093.7472	247.24015