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REGULARITY THEOREM FOR FUNCTIONS THAT ARE EXTREMAL TO PALEY INEQUALITY

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ABSTRACT: In this paper we study the asymptotic behavior of functions that are extremal to the inequality introduced by Paley (1932) via a normal family of subharmonic functions.

Key words/phrases: Subharmonic, characteristic function, lower order, Pólya peaks, *-function.

INTRODUCTION

Let u be a subharmonc function in the complex plane. We set

$$B(r,u) = \sup_{|Z|=r} u(z), \quad u^+(z) = \max(u(z),0).$$

$$N(r,u) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta$$

and define the Nevanlinna characteristic of u by $T(r) = T(r,u) = N(r,u^+)$.

The order ρ of u is by definition

$$\rho = \limsup_{r\to\infty}\frac{\log T(r,u)}{\log r}$$

Paley's inequality asserts that

$$\underline{\lim}\ \frac{B(r,u)}{T(r,u)} \le \frac{\pi\rho}{\sin\pi\rho} \qquad (1)$$

provided $0 < \rho \le \frac{1}{2}$.

Inequality (1) was conjectured by Paley (1932) in the case where u=log|f(z)| for entire function f. Govorov (1969) proved Paley's conjecture. For a proof of (1) for general subharmonic u, see Essen (1975). The function

$$u(re^{i\theta}) = \frac{\pi\rho r^{\rho}\cos\rho\theta}{\sin\rho\pi}, \ |\theta| \le \pi \tag{2}$$

in a subharmonic function which is extremal for (1) with

$$T(r,u) = r^{\rho}$$

and

$$B(r,u) = \frac{\pi\rho r^{\rho}}{\sin\pi\rho} \ .$$

Indeed all subharmonic functions extremal for (1), *i.e.*, for which equality holds in (1) behave asymptotically as (rotations of) the subharmonc function in (2). We shall prove,

Theorem 1: Let u be a subharmonc function of order ρ, $0 < \rho \le \frac{1}{2}$. If

$$\overline{\lim} \frac{T(r,u)}{B(r,u)} = \frac{\sin\pi\rho}{\pi\rho} \tag{3}$$

Then

i) $T(r,u) = r^{\rho} L(r)$

ii) $B(r,u) \sim \frac{\pi\rho}{\sin\pi\rho} T(r,u)$, $(r \to \infty, r \in G)$

iii) $N(r,u) \sim T(r,u)$, $(r \to \infty, r \in G)$,

where L(r) varies slowly in a set G of logarithmic density one, i.e.

$$\lim_{\substack{r\to\infty \\ r\in G}} \frac{L(\sigma r)}{L(r)} = 1, \quad (0 < \sigma < \infty)$$

holds (uniformly for σ in any interval $A^{-1} \le \sigma A$, $A > 1$) with

$$G = \bigcup_{n=1}^{\infty}[a_n, b_n], \left(a_n \to \infty, \frac{b_n}{a_n} \to \infty\right) \tag{4}$$

satisfying

$$\int_{G\cap[1,r]} t^{-1}dt \sim \log r, \quad (r \to \infty).$$

Moreover, for any sequence $\{r_n\} \subseteq G$ ($r_n \to \infty$) there is a subsequence of positive integers $I = \{n_k\}$ and a real $\alpha \in [-\pi,\pi]$ such that for almost all z in the set $\{re^{i\theta}, |\theta - \alpha| < \pi\}$ with respect to the Lebesgue measure of the plane we have

$$\text{iv)} \qquad \lim_{\substack{n\to\infty \\ n\in I}} \frac{u(r_n z)}{T(r_n,u)} = \frac{\pi\rho r^\rho}{\sin \pi\rho} \cos \rho(\theta - \alpha), \quad |\theta - \alpha| \leq \pi, \quad z = re^{i\theta}$$

SOME FACTS

Let u be a subharmonic function in $\mathbb{C}$ of order ρ, $0 \leq \rho < \infty$. By a sequence of Pólya peaks for $T(r,u)$ of order ρ we mean a sequence $\{r_n\}$ of positive numbers such that $T(t,u) \leq (1 + \varepsilon_n) \left(\frac{1}{r_n}\right)^\rho T(r_n,u)$ holds for some sequence $\varepsilon_n \to 0$, $r\ \varepsilon_n \to \infty$ and for all t, such that $\epsilon_n r_n \leq t \leq r_n/\varepsilon_n$. If ρ is finite $T(r,u)$ has a sequence of Pòlya peaks of order ρ (for a proof see Edrei (1965)). We also need the *-function of u introduced by Baernstein (1974) which is defined by

$$u^*(re^{i\theta}) = \frac{1}{2\pi} \sup_{|E|=2\theta} \int_E u(re^{\Phi})d\Phi, \ (0 < r < \infty, 0 \leq \infty \leq \pi), \tag{5}$$

where $E \subseteq [-\pi, \pi]$ and $|E|$ = Lebesgue measure of the set E. It is shown that u^* is subharmonic in the upper half plane, π^+ and is continuous in the closure of π^+ except possibly at the origin. It is also proved that if the function $\theta \to \tilde{u}(re^{i\theta})$, for fixed $r > 0$ and $|\theta| \leq \pi$, is the symmetric decreasing rearrangement of $u(re^{i\theta})$, then

$$u^*(re^{i\theta}) = \frac{1}{2\pi} \int_{-\theta}^{\theta} \tilde{u}(re^{i\phi})d\phi \quad (0 \leq \theta \leq \pi).$$

u^* satisfies (see also Hayman (1989), chap. 9, especially p. 712).

$$T(r,u) = \max_{0\leq\theta\leq\pi} u^*(re^{i\theta}),\ u^*(r) = 0 \tag{6}$$

and

$$B(r,u) = \frac{\pi\partial u^*}{\partial\theta}(re^{i\theta})\,|_{\theta=0} = \tilde{u}(r).$$

Proof of Theorem 1: First we shall establish assertions (i) and (ii) of Theorem 1. We use the well-known result due to Petrenko (1969) given below by Lemma 1.

Lemma 1. Suppose u is subharmonic in $\mathbb{C}$. Fix γ, $0 < \gamma \le 1$, and let

$$k(t,\gamma) = \frac{\gamma^{-2} t^{1/\gamma}}{\left(t^{1/\gamma}+1\right)^2}.$$

Then

$$B(r,u) \le \int_0^R u(te^{i\pi\gamma})\, k\,(r/t,\gamma)\, \frac{dt}{t} + c\left(\frac{r}{R}\right)^{1/\gamma} T(2R,u), \quad \left(0 < r < \frac{R}{2}\right), \tag{7}$$

for an absolute constant c.

Besides Petrenko's (1969) proofs for Lemma 1 are given in Essen (1975) and in Edrei and Fuchs (1976, Lemma 11.1) where it was shown that

$$\hat{k}\,(s,\gamma) = \int_0^\infty k(t,\gamma) \frac{dt}{t^{1+s}} = \frac{\pi s}{\sin(\pi\gamma s)}, \; \left(0 < r \frac{1}{\gamma}\right).$$

We apply Lemma 1 with $\gamma = 1$ together with the hypotheses of Theorem 1 to obtain (on letting $R \to \infty$) in (7)

$$T(r,u) \le \left(\frac{\sin \pi\rho}{\pi\rho} + o(1)\right) \int_0^\infty T(t,u) k(r/t,\gamma) \frac{dt}{t}, \; (r \to \infty) \tag{8}$$

$$= \left(\frac{\sin \pi\rho}{\pi\rho} + o(1)\right) T(r,u)^* \, k(r,\gamma).$$

Since $\hat{k}(\rho,\gamma) = \dfrac{\pi\rho}{\sin \pi\rho}$ and T(r,u) is increasing it follows from the Drasin and Shea (1976) Tauberian theorem that there is a set G of the from (4) such that

$$T(r,u) = r^\rho L(r) \tag{9}$$

where L(r) varies slowly in G, and further

$$T(r,u)^* \, k(r,\gamma) = \left(\frac{\pi\rho}{\sin \pi\rho} + o(1)\right) T(r,u), \quad (r \to \infty \in G). \tag{10}$$

Thus from (7), (10) and the hypotheses of Theorem 1 we conclude that

$$\lim_{\substack{r\to\infty \\ r\in G}} \frac{B(r,u)}{T(r,u)} = \frac{\pi\rho}{\sin \pi\rho}. \tag{11}$$

This proves assertions (i) and (ii) of Theorem 1. It also follows from (9) that any sequence $\{r_n\} \subseteq G$, $r_n \to \infty$, is a sequence of Polya peaks for T(r,u) of order ρ (see Edrei (1969) Lemma 4). We proceed to prove Theorem 1.

Let $\{r_n\}$ be any sequence of Polya peaks for T(r,u) of order ρ, $0 \leq \rho < \infty$. Then the sequence

$$u_n(z) = \frac{u(r_n z)}{T(r_n u)} \quad (z \in \mathbb{C}, \quad n = 1, 2, 3, \ldots) \tag{12}$$

forms a normal family of subharmonic functions in the sense of Anderson and Baernstein (1978), *i.e.*, there is a subharmonic function v is $\mathbb{C}$ and a subsequence $I = \{n_k\}$ of positive integers such that

$$\text{a)} \quad \lim_{\substack{n\to\infty \\ n \in I}} \int_0^{2\pi} | u_n(re^{i\theta}) - v(re^{i\theta}) | \, d\theta = 0, \quad (0 < r < \infty) \tag{13}$$

$$\text{b)} \quad \lim_{\substack{n\to\infty \\ n \in I}} T(r,u_n) = \lim_{\substack{n\to\infty \\ n \in I}} \frac{T(rr_n,u)}{T(r_n,u)} = T(r,v) \leq r^\rho, \; (0 < r < \infty).$$

Thus if u satisfies the hypotheses of Theorem 1 then from (9) and (13) we have

$$T(r,v) = r^\rho. \tag{14}$$

Indeed we prove,

Theorem 2: Let u be a subharmonic function that satisfies the hypotheses of Theorem 1 and $\{r_n\}$ a sequence of Pólya peaks of T(r,u) of order ρ, $0<\rho\leq\frac{1}{2}$. If v is a subharmonic function that satisfies (13) corresponding to u and the sequence $\{r_n\}$, then

a) $T(r,v) = r^\rho = N(r,v)$

b) $v(re^{i\theta}) = \dfrac{\pi\rho r^\rho}{\sin \pi\rho} \cos \rho(\theta - \alpha)$, $|\theta - \alpha| \leq \pi$ for some $\alpha \in [-\pi,\pi]$.

It is clear that assertions (iii) and (iv) of Theorem 1 follow from Theorem 2 and (13). We first establish,

Lemma 2. Suppose u satisfies the hypotheses of Theorem 1 and $\{r_n\}$ is a sequence of Pólya peaks of T(r,u) of order ρ, $0 < \rho \le \frac{1}{2}$. If v is a subharmonic function that satisfies (13) corresponding to u and the sequence $\{r_n\}$ then v* is harmonic in the upper half plane and

$$v^*(re^{i\theta}) = r^\rho \frac{\sin \rho^\theta}{\sin \pi\rho} \qquad (0 \le \theta \le \pi)$$

Proof. First we prove $B(r,v) = \frac{\pi\rho r^\rho}{\sin \pi\rho}$. Since $v^*(re^{i\pi}) \le T(r,v) = r^\rho$ and $v^*(r)=0$ by (6) and (14), we have by Phragme'n – Lindeöf principle

$$v^*(re^{i\theta}) \le \frac{r^\rho \sin \rho^\theta \theta}{\sin \pi\rho} \qquad (0 \le \theta \le \pi) \tag{15}$$

which implies $\frac{\partial v^*}{\partial \theta}(re^{i\theta})\,|_{\theta=0} \le \frac{\rho r^\rho}{\sin \pi\rho}$. Thus it follows from (6) that

$$B(r,v) \le \frac{\pi\rho r^\rho}{\sin \pi\rho}$$

To prove the reverse inequality we let r > 0 and choose $\alpha_n \in [-\pi,\pi)$, $(n \in I = \{n_k\})$ such that $B(r,u_n) = u_n(re^{i\alpha_n})$, where u_n is defined by (12).

Assume $\alpha_n \to \alpha_0$ as $n \to \infty$ $(n \in I)$. Then for any s > r, we have

$$B(r,u_n) \le \frac{1}{2\pi} \int_0^{2\pi} u_n\,(se^{i\theta}) p_{r/s}(\theta - \alpha_n) d\theta. \tag{16}$$

where $p_r(\theta)$ is the Poisson kernel. By dominated convergence theorem and by (13) it follows (letting $n \to \infty$, $n \in I'$ in (16)) that

$$\frac{\pi\rho r^\rho}{\sin \pi\rho} \le \frac{1}{2\pi} \int_0^{2\pi} v(se^{i\theta})\, p_{r/s}(\theta - \alpha_n) d\theta \le B(s,v). \tag{17}$$

Since (17) holds for any s > r and B(s,v) is a continuous function of s, we have, letting $s \to r$ in (17), $\frac{\pi\rho r^\rho}{\sin \pi\rho} \le B(r,v)$. Thus $B(r,v) = \frac{\pi\rho r^\rho}{\sin \pi\rho}$. To prove the lemma we apply (7) to v with $\gamma = \frac{\alpha}{\pi}$ $(0 < \alpha < \pi)$, and using (16) we get

$$\frac{\pi\rho r^{\rho}}{\sin\pi\rho} = B(r,v) \le \int_0^{\infty} v^*(te^{i\alpha})\, k(r/t,\gamma)\frac{dt}{t} \le \frac{\sin\rho\alpha}{\sin\pi\rho}\int_0^{\infty} t^{\rho} k(\frac{r}{t},\gamma)\frac{dt}{t} = \frac{\pi\rho r^{\rho}}{\sin\pi\rho}$$

which implies $v^*(te^{i\alpha}) = t^{\rho}\frac{\sin\rho\alpha}{\sin\pi\rho}$. Thus applying the maximum principle to (15) we conclude that $v^*(re^{i\theta}) = \frac{r^{\rho}\sin\rho^{\theta}}{\sin\pi\rho}$, $(0 \le \theta \le \pi)$.

To complete the proof of Theorem 2, let $\widetilde{v}$ be the symmetric decreasing rearrangement of v, so that

$$v^*(re^{i\theta}) = \frac{1}{2\pi}\int_{-\theta}^{\theta}\widetilde{v}\,(re^{i\phi}),\ (0 \le \theta \le \pi). \tag{18}$$

Since by Lemma 2 $v^*(z)$ is harmonic in $\{z: 0 < \arg(z) < \pi\}$, it follows from (18) that

$$0 = \Delta v^*(re^{i\theta}) = \frac{1}{2\pi}\int_{-\theta}^{\theta}\Delta\widetilde{v}\,(re^{i\phi})d\phi,\ (0 < \theta < \pi).$$

Thus $\widetilde{v}$ is harmonic in $\{re^{i\theta} : r > 0, |\theta| < \pi\}$. Let $r_1 > 0$ and $\alpha \in [-\pi, \pi]$ be such that $v(r_1e^{i\alpha}) = B(r_1,v) = \widetilde{v}(r_1)$. Then a comparison of v^* with the subharmonic function

$$v_{\alpha}(re^{i\theta}) = \frac{1}{2\pi}\int_{-\theta+\alpha}^{\theta+\alpha} v(re^{i\phi})d\phi$$

shows that (Essén and Shea , 1978/79)

$$v(ze^{i\alpha}) = \widetilde{v}\,(z).\ (|\arg(z)| < \pi). \tag{19}$$

By Lemma 2, (18) and (19) we get

$$v(re^{i\theta}) = \frac{\pi\rho r^{\rho}}{\sin\pi\rho}\cos\rho(\theta-\alpha),\quad (|\theta-\alpha| \le \pi). \tag{20}$$

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