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APPLICATION OF NORMAL FAMILY OF δ-SUBHARMONIC FUNCTIONS TO THE EDREI-FUCHS ELLIPSE THEOREM

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ABSTRACT: In this paper we study the asymptotic behaviour *of* functions extremal for the well known inequality introduced by Edrei-Fuchs (called the Ellipse Theorem) by considering a normal family of δ-subharmonic functions. This approach allows us to describe precisely the prototype of all functions extremal for the Edrei-Fuchs ellipse theorem. Indeed, it turns out that the functions which are extremal for the Edrei-Fuchs inequality are also extremal for the inequality introduced by Essèn-Rossi-Shea.

Key words/phrases: δ-subharmonic, characteristic function, deficiency, lower order, Pólya peaks,*-function

INTRODUCTION

Let f be a meromorphic function in the complex plane $\mathbb{C}$ of lower order λ, $0 < \lambda < 1$. We assume the reader is familiar with the notations $T(r,f)$, $N(r,f)$, $\delta(\infty,f)$ involved in the theory of Nevanlinna. We set:

$$a = 1 - \delta(0,f), \qquad b = 1 - \delta(\infty,f).$$

The Edrei-Fuchs Ellipse Theorem (Edrei *et al.*, 1960) states that

$$a^2 + b^2 - 2ab\cos\lambda\pi \geq \sin^2\lambda\pi. \tag{1}$$

Furthermore $a \leq \cos\lambda\pi$ implies $b = 1$ and $b \leq \cos\lambda\pi$ implies $a = 1$.

Besides (Edrei *et al.*, 1960) proofs for (1) are given in Edrei (1969) and in (Hayman, 1989, Ch. 9) using the spread theorem due to Baernstein (1973). In addition, the behaviour of the extremal functions, *i.e.*, one for which equality holds in (1) has been discussed by Edrei (1969). In this paper we give a proof of (1) and also describe the behaviour of the extremal functions using a "normal family" of δ-subharmonic functions (functions representable as a difference of subharmonic functions) in the sense of Anderson and Baernstein (1978). The approach has also been used to describe the asymptotic behaviour of a certain class of extremal functions, such as the well known inequality, Paley inequality (Hayman, 1989, Ch. 9).

Let $u = u_1 - u_2$ be a δ-subharmonic function in the complex plane, where u_i (i= 1,2) is a subharmonic function. If $u = \log|f|$, f meromorphic, then u is a δ-subharmonic function. Any $u \in C^2(\mathbb{C})$ is δ-subharmonic (Arsove, 1953). We write

$$N(r,u) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta \qquad (0<r<\infty), \qquad u^+ = \max(u.0)$$

and define the Nevanlinna characteristic of u by

$$T(r) = T(r,u) = N(r,u^+) + N(r,u_2).$$

Clearly, T(r,u) depends on the particular decomposition $u_1 - u_2$ as well as on u, but one can ignore such an ambiguity, for discussion see (Hayman, 1989, Ch. 9).

The lower order λ of u is defined by

$$\lambda = \liminf_{r\to\infty} \frac{\log T(r)}{\log r}$$

and the Nevanlinna deficiency of u

$$\delta(\infty,u) = \liminf_{r\to\infty} \frac{N(r,u^+)}{T(r,u)} = 1 - \liminf_{r\to\infty} \frac{N(r,u_2)}{T(r,u)}$$

Throughout this paper we shall write

$$a = 1 - \delta(0, u), \text{ where } \delta(0, u) = \delta(\infty, -u)$$
$$b = 1 - \delta(\infty, u). \qquad (2)$$

Clearly $0 \le 1 \le a,\ b \le 1.$

STATEMENT OF RESULTS

A sequence $\{r_n\}$ of positive numbers is a sequence of Pólya peaks for T(r,u) of order λ, $0 \leq \lambda < \infty$, if there is a sequence $\{\epsilon_n\}$, $\epsilon_n > 0$, such that as $n \to \infty$, $\epsilon_n \to 0$ and $\epsilon_n r_n \to \infty$ and such that $\epsilon_n r_n \leq t \leq r_n/\epsilon_n$ implies

$$T(t,u_n) \leq (1+\epsilon_n)\left(\frac{t}{r_n}\right)^{\lambda} T(r_n, u).$$

If λ is finite T(r,u) has a sequence of Pólya peaks of order λ (for a proof see (Edrei, 1965)). Let us state a special case of the "normal family" theorem due to Baernstein (1973).

Theorem A

Let $u = u_1 - u_2$ be a δ-subharmonic function of lower order λ, $0 \leq \lambda < \infty$, and let $\{r_n\}$ be a sequence of Pólya peaks for T(r,u) of order λ. For each n = 1, 2, 3 ... let

$$u_n(z) = \frac{u(r_n z)}{T(r_n, u)} = u_n^1(z) - u_n^2(z) \qquad (z \in \mathbb{C}) \tag{3}$$

where $u_n^i(z) = \dfrac{u_i(r_n z)}{T(r_n, u)}$ $(i=1,2)$.

Then there is a δ-subharmonic function $v = v_1 - v_2$ in $\mathbb{C}$ and a subsequence $I = (n_k)$ of positive integers such that (as $n \to \infty$, $n \in I$) the following holds:

a) $\lim\limits_{n\to\infty} N(r, |u_n - v|) = 0$ $(0<r<\infty)$,

b) $\lim\limits_{n\to\infty} T(r, u_n) = T(r, v) \leq r^{\lambda}$ $(0<r<\infty)$,

c) $\lim\limits_{n\to\infty} N(r, u_n^1) = N(r, v_1) \leq (1 - \delta(0,u))r^{\lambda}$ $(0<r<\infty)$,

d) $\lim\limits_{n\to\infty} N(r, u_n^2) = N(r, v_2) \leq (1 - \delta(\infty,u))r^{\lambda}$ $(0<r<\infty)$, We notice from (3) and (b) that $T(1,v) = 1$. Since a subsequence of $\{r_n\}$ is a sequence of Pólya peaks for $T(r,u)$, we consider the class of all δ-subharmonic functions $v=v_1-v_2$ such that for some subsequence of $\{r_n\}$ assertions (a) to (d) of Theorem A hold. We denote this class by $S(u,\{r_n\})$. We also write $|E|$ for the Lebesgue measure of the set E.

We shall prove

Theorem 1

Let $u = u_1 - u_2$ be a δ-subharmonic function of lower order λ, $0 < \lambda < 1$, and $\{r_n\}$ a sequence of Pólya peaks for $T(r,u)$ of order λ. Then

(i) $a^2+b^2-2ab\cos\lambda\pi \geq \sin^2\lambda\pi.$

Furthermore $a \leq \cos\lambda\pi$ implies $b = 1$ and $b \leq \cos\lambda\pi$ implies $a = 1$, where a and b are defined in (2).

(ii) Let h be any nonnegative function defined for $r > 0$ and satisfying $h(r) = 0(\log r)$ as $r\to\infty$ and consider the sequence:

$$\beta_n=\beta_n(h)=\frac{1}{2}|\{\theta \in[-\pi,\pi]: u(r_n e^{i\theta})>h(r_n)\}| \quad (n = 1,2,3,...) \tag{4}$$

If equality holds in (i) with $a < 1$ and $b < 1$, then

(a′) $$\beta = \lim_{n\to\infty} \beta_n = \frac{1}{\lambda}\cos^{-1}(b), \qquad (0 \leq \beta \leq \frac{\pi}{2})$$

and we have $b = \cos\lambda\beta$ and $a = \cos\lambda(\pi - \beta)$.

Furthermore for each $v = v_1 - v_2 \in S(u,\{r_n\})$ we have

(b′) $T(r,v) = r^\lambda$, $N(r,v_1) = ar^\lambda$, $N(r,v_2) = br^\lambda$, and

(c′) $v(re^{i\theta}) = \pi\lambda r^\lambda \sin\lambda(\beta-|\theta-\alpha|)$ for some $\alpha\in[-\pi,\pi]$ and $|\theta - \alpha| \leq \pi$.

All the assertions of Theorem 1 except part (c′) are well known for u = log |f|, where f is a meromorphic function (see Theorem 1, Edrei, 1969). However, the Ellipse Theorem, that is assertion (i) of Theorem 1 is already known for δ-subharmonic function. Thus Theorem 1, apart from generalising Edrei's result to arbitrary δ-subharmonic functions it describes precisely the functions which are extremal for the Ellipse inequality. The function v in (c′) of Theorem 1 is a δ-subharmonic function (usually called a harmonic spline (see Hayman, 1989, p. 739) extremal for (i). Indeed the function v in (c′) is a typical example extremal for the inequality introduced by Essén, Rossi and Shea (see Theorem 1, Essén *et al*, 1992). Thus, in some sense the functions extremal for (i) of Theorem 1 are also extremal for the inequality of Essen-Rossi-Shea (Theorem 1, Essén *et al*, 1992). Also Theorem A (a) and Theorem 1 (c′) imply that there is a subsequence $I = \{n_k\}$ of positive integers and a real $\alpha \in [-\pi, \pi]$ (depending on v) such that for almost all $z (z = re^{i\theta})$,
$u_n(re^{i\theta}) \to v(re^{i\theta})$ as $n \to \infty$, $n \in I$.

SOME FACTS

One of our main tools to prove Theorem 1 is the *-function introduced by Baernstein (1974). Let $u = u_1 - u_2$ be a δ-subharmonic function in $\mathbb{C}$. The *-function of u, denoted by u*, is defined

$$u^*(re^{i\theta}) = \frac{1}{2\pi} \sup \int_E u(re^{i\phi})d\phi + N(r, u_2) \qquad (0 \le \theta \le \pi, \quad r > 0)$$

where the supremum is taken over all sets $E \subseteq [-\pi, \pi]$ with $|E| = 2\theta$. In (Bernstien, 1974) it is proved that u* is subharmonic in the upper half plane, π^+ and continuous in the closure of π^+ except possibly at z = O. Also in (Baernstein, 1974) it is proved that for each $r > 0$ and $0 \le \theta \le \pi$ there is a set $A \subseteq [-\pi, \pi]$ with $|A| = 2\theta$ such that

$$\sup_{|E| = 2\theta} \int_E u(re^{i\phi})d\phi = \int_A u(re^{i\phi})d\phi = \int_{-\theta}^{\theta} \tilde{u}(re^{i\phi})d\phi,$$

where $\theta \to \tilde{u}(re^{i\theta})$ is a symmetric decreasing rearrangement of $u(re^{i\theta})$ on $[-\pi,\pi]$. Further properties of u* (see also Hayman, 1989, Ch. 9) needed here:

$$T(r,u) = \max_{\theta} u^*(re^{i\theta}),\ u^*(r) = N(r,u_2),\ u^*(-r) = N(r,u_1), \tag{5}$$

$$B(r,u) = \sup_{|z|=r} u(z) = \pi \frac{\partial}{\partial\theta} u^*(re^{i\theta})|_{\theta} = 0.$$

Proof of Theorem 1

Lemma 1

Let $u = u_1 - u_2$ be a δ-subharmonic function of lower order λ, $0 < \lambda < \infty$ and let $\{r_n\}$ be a sequence of Pólya peaks for T(r,u) of order λ. If β is any accumulation point of $\{\beta_n\}$, where β_n is defined in (4), then there is a δ-subharmonic function $v = v_1 - v_2 \in s(u, \{r_n\})$ such that $v^*(e^{i\beta}) = 1$. Conversely if $v = v_1 - v_2 \in s(u,\{r_n\})$ then $v^*(e^{i\beta}) = 1$ for some accumulation point β of $\{\beta_n\}$.

Proof

Consider the δ-subharmonic function

$$\begin{aligned} w_n(re^{i\theta}) &= u_n^1(re^{i\theta}) - [u_n^2(re^{i\theta}) + \frac{h(r_n)}{T(r_n,u)}], \qquad (n = 1,2,3,\ldots) \\ &= w_n^1(re^{i\theta}) - w_n^2(re^{i\theta}), \end{aligned}$$

where $u_n^i (i = 1,2)$ is defined in (3), h is as in Theorem 1, and $w_n^1 = u_n^1$ and $w_n^2 = u_n^2 + \frac{h(r_n)}{T(r_n,u)}$.

It is easy to check that for each $v = v_1 - v_2 \in S(u, \{r_n\})$ there is a subsequence of $\{w_n\}$ such that all assertions of Theorem A hold with w_n instead of u_n. Let β be any accumulation point of $\{\beta_n\}$. Assume $\beta_n \to \beta$. Let $v = v_1 - v_2 \in S(u, \{r_n\})$. Then there is subsequence $I = \{n_k\}$ of positive integers such that the function v and the subsequence $\{w_n\}$, $n \in I$, satisfy all assertions of Theorem A. The definition of β_n

and w_n shows that $w_n^*(e^{i\beta_n})=1$ for all n. Let $A_n \subseteq [-\pi, \pi]$ with $|A_n| = 2\beta_n$ and such that

$$1=\frac{1}{2\pi}\int_{A_n} w_n(e^{i\theta})d\theta + N(1, w_n^2).$$

Now applying Theorem A we obtain

$$1=\frac{1}{2\pi}\int_{A_n} \nu(e^{i\theta})d\theta + N(1,\nu_2)+o(1) \quad (n\to\infty,\ n\in I)$$

$$\le \nu^*(e^{i\beta_n})+ o(1) \quad (n\to\infty,\ n\in I)$$

The continuity of ν^* implies $1 \le \nu^*(e^{i\beta})$. Since $\nu^*(e^{i\beta}) \le T(1, \nu) = 1$ we conclude that $\nu^*(e^{i\beta}) = 1$. The converse is similarly proved.

Lemma 2

Let $u = u_1 - u_2$ be a δ-subharmonic function of lower order λ, $0 <\lambda < 1$, and $\{r_n\}$ a sequence of Pólya peaks of order λ for $T(r,u)$. If β is any accumulation point of $\{\beta_n\}$ where β_n is defined in (4) of Thereom 1, then

$$\sin\lambda\pi \le a\sin\lambda\beta + b\sin\lambda(\pi-\beta) \tag{6}$$

where the numbers a and b are defined in (2).

Proof

Let $I=\{n_k\}$ be a sequence of positive integers such that

$$\lim_{n\to\infty} \beta_n=\beta, \qquad (n\in I).$$

Lemma 1 shows that for any $\nu = \nu_1 - \nu_2 \in S(r, \{r_n\}), n \in I$, $\nu^*(e^{i\beta}) = 1$. Put

$$S(\theta)=\frac{a\sin\lambda\theta+b\sin\lambda(\pi-\theta)}{\sin\lambda\pi} \quad (0\le\theta\le\pi,\ 0<\lambda<1), \tag{7}$$

and define

$$w(re^{i\theta}) = v^*(re^{i\theta}) - r^{\lambda}S(\theta) \qquad (0 \leq \theta \leq \pi, \quad r > 0).$$

The function w is subharmonic on the upper half plane, π^+. Using Theorem A and (5) we obtain $w(r) \leq 0$, $w(-r) \leq 0$ and $w(re^{i\theta}) \leq r^{\lambda}$. Since $\lambda\pi < \lambda$ we conclude by Phragmén-Lindelöf principle that $w(re^{i\theta}) \leq 0$ Thus

$$v^*(re^{i\theta}) \leq r^{\lambda}S(\theta) \qquad (0 \leq \theta \leq \pi, \quad r > 0). \tag{8}$$

The Lemma is now proved by setting r = 1 and $\theta = \beta$ in (8).

We proceed to prove Theorem 1. Using Cauchy - Schwarz inequality (6) implies

$$\begin{aligned} \sin\lambda\pi &\leq (a - b\cos\lambda\pi)\sin\lambda\beta + b\sin\lambda\pi \cos\lambda\beta \\ &\leq \left[(a - b\cos\lambda\pi)^2 + b^2\sin^2\lambda\pi\right]^{1/2} \\ &= \left[a^2 + b^2 - 2ab\cos\lambda\pi\right]^{1/2}. \end{aligned} \tag{9}$$

This proves assertion (i) of Theorem 1. For the proof of the implications or $a \leq \cos\lambda\pi$ or $b \leq \cos\lambda\beta\pi$ and assertion ii (a') of Theorem 1 see section 5 and 6 of Theorem 1 (Edrei, 1969).

Proof of assertions ii (b') and ii (c') of Theorem 1

The assumptions a < 1 and b < 1 together with Lemma 2 imply $\beta \neq 0, \pi$, where β is any accumulation point of $\{\beta_n\}$ defined in (4). The hypotheses that equality holds in (i) of Theorem 1 implies equality in (9) and hence in (6). Thus $v^*(e^{i\beta}) = 1 = S(\beta)$, for any $v \in S(u, \{r_n\})$. Since $0 < \beta < \pi$, the maximum principle for subharmonic function and (8) imply

$$v^*(re^{i\theta}) = r^{\lambda}s(\theta) \qquad (0 \leq \theta \leq \pi). \tag{10}$$

Consequently v^* is harmonic in π^+ and by (4) we have

$$N(r, v_1) = v^*(re^{i\pi}) = ar^{\lambda}, \quad N(r, v_2) = v^*(r) = br^{\lambda} \text{ and } T(r, v) = r^{\lambda}. \tag{11}$$

This proves assertion ii (b') of Theorem 1. The relations in (11) allow us to define $v_1(0) = 0 = v_2(0)$. We proceed to find the extremal functions. We prove Lemma 3.

Lemma 3

Let u = u_1 - u_2 be a δ-subharmonic function in $\mathbb{C}$ of order λ, $0<\lambda<1$ and $u_1(0) = 0 = u_2(0)$. if u* is harmonic in the upper half plane then

$$u(z)=\int_0^\infty \log\left|1+\frac{ze^{i\alpha}}{t}\right|dn_1(t) - \int_0^\infty \log\left|1-\frac{ze^{i\alpha}}{t}\right|dn_2(t) \qquad (12)$$

for some $\alpha \in [-\pi,\pi]$, where $n_i(t)=\mu_i(|z| \le t)$ and μ_i is the Riesz measure of u_i $(i=1,2)$.

Proof

Since $0 < \lambda < 1$ we have the representation

$$u(z)=\int_{\mathbb{C}} \log\left|1-\frac{z}{\zeta}\right|d\mu_1(\zeta)-\int_{\mathbb{C}} \log\left|1-\frac{z}{\zeta}\right|d\mu_2(\zeta). \qquad (13)$$

Consider the δ-subharmonic function

$$w(z)=\int_{\mathbb{C}} \log\left|1+\frac{z}{|\zeta|}\right|d\mu_1(\zeta)-\int_{\mathbb{C}} \log\left|1-\frac{z}{|\zeta|}\right|d\mu_2(\zeta) \qquad (14)$$
$$=w_1(z) - w_2(z).$$

It is easy to check that w* is harmonic in π^+ and that $N(r,u_i) = N(r,w_i)$, $(i = 1,2)$. Thus using (4) and the Phragmén-Lindelöf principle, $\lambda\pi<\pi$ we conclude that $w^*(re^{i\theta}) = u^*(re^{i\theta})$, $0 \le \theta \le \pi$, $r>0$). Again by (4) and (14) we have B(r,u) = B(r,w) = w(r). Let B(r,u) $=u(re^{i\alpha})$ for some $\alpha \in [-\pi,\pi]$. Now the Lemma follows using (12), (13) together with $u(re^{i\alpha}) = w(r)$.

To prove ii (c′) of Theorem 1, we let $v=v_1-v_2 \in S(u, \{r_n\})$, where u is extremal for (i) of Theorem 1. Then by (10) v satisfies the hypotheses of Lemma 3. From Jensen's formula and (11) we have $n_1(t)=a\lambda t^\lambda$ and $n_2(t)=b\lambda t^\lambda$. Thus using (12) and applying the residue theorem of calculus yields assertion ii (c′).

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