Students' Difficulties and Misconceptions in Learning Concepts of Limit, Continuity and Derivative

Solomon Areaya* and Ashebbir Sidelil **

Abstract: This study aimed at examining students' difficulties and misconceptions in learning concepts of calculus at preparatory secondary schools of Dire Dawa city. Accordingly, students' conception of concepts in calculus, students' misconceptions, and factors influencing the teaching-learning of concepts were surveyed. Descriptive survey approach was used as a research method for the study. Achievement test was the prime instrument used for gathering the necessary data. One hundred thirty-five students were involved in the study. The study result indicated gaps between the aspiration of the mathematics syllabi prepared by the Ethiopian Federal Ministry of Education and students' actual achievement. It also indicated the presence of misconceptions, on the part of students, regarding the basic concepts of calculus.

Key Words: Conceptual knowledge, Learning calculus, Limit Concept, Misconception, Procedural knowledge

* Assistant Professor, Department of Teachers' Professional Development and Curriculum Studies, Addis Ababa University. Email: solomonareaya@yahoo.com

**Lecturer, Technical Vocational Education and Training (TVET) College, DireDawa. Email: ashebbir14@yahoo.com

Introduction

Preliminary knowledge of calculus is a pre-requisite for most physical science fields at tertiary education. Hence, it is crucial to create appropriate context and resource for effective teaching and learning of mathematics in general and the concept of calculus in particular at preparatory secondary schools. Concepts serve as a base in teaching and learning mathematics. The understanding of subsequent concepts is hardly possible if pre-requisite concepts are not clearly established. For example, a student cannot understand the concept of derivatives before understanding the concept of limits. Experience and observations in Ethiopian secondary schools indicate that introductory calculus is often challenging for both students as well as teachers. There are terminologies such as: *tends to, approaches but never touch, as small as we please, infinity,* which seem are source of confusion and cause for misconception on the part of good numbers of students.

This study has a potential benefit for mathematic teachers in that it provides them information about their students' possible misconceptions. This in turn could enable them to make the necessary intervention towards addressing the observed as well as anticipated problems. Although the teaching and learning of concepts in calculus could be researched from different dimensions this study, however, is delimited in assessing the difficulties, challenges, and misconceptions of learning concepts of calculus. The *limit* concept, limit of a sequence, limit of functions, continuity, and derivative are the themes which define the scope of this study.

Statements of the Problem

As mathematics educators and/or teachers, we were interested to undertake a study aimed at understanding and illuminating problems and difficulties that students encounter in learning mathematics in general and concepts of calculus in particular. Hence, the aim of this study was to identify the difficulties, challenges, and misconceptions in learning concepts of calculus (limit, continuity, and derivative) in preparatory secondary schools of Dire

Dawa city. Accordingly, the study attempted to answer the following research questions: 1) what misconceptions do students form in learning concepts of calculus? 2) what are the difficulties and challenges of students in learning concepts of calculus?

The Concept of a Limit

Mathematicians and mathematics educators agree that the term ***limit*** is the most fundamental concept in calculus. Even major concept of calculus namely, derivative, continuity, integral, convergence or divergence are defined in terms of limits. However, a complete understanding of the limit concept among students is comparatively rare. Moreover, many of the difficulties encountered by students in dealing with other concepts in calculus are related to their difficulties in understanding the concept of a limit. The concept of 'limit'' could be described either explicitly or implicitly. An informal (implicit) definition of limit presented in grade 12 students textbook is that (MOE, 2006_b, p. 122):

> Let y=f(x) be a function. Suppose that **a** and **L** are real numbers such that as **x** gets closer and closer to **a** but not equal to **a**, f(x) gets closer and closer to **L**, then we say that the **limit of f(x) as x approaches *a* is L,** and we write $\lim_{x\to a} f(x) = L$.

The formal (explicit) definition of limit is that (Ellis and Gulick, 1993):

> Let **f** be a function defined at each point of some open interval containing **a**, except possibly at **a** itself. Then **a number L is the limit of f(x) as x approaches a** (or L is the limit of f at a) if for every number $\varepsilon < 0$ there is a number $\delta > 0$ such that if $0 < |x - a| < \sigma$, then $|f(x) - L| < \varepsilon$.

The language used, with terms like '***tends to***' or '***approaches*** ' or ' ***has a limit*** ' all suggest that the expression gets closer to the limit but not equal to it . The calculation of a limit involves an infinite number of computations.

Once a calculation involves an infinite number of steps; it can only be understood through a process conception (Cottril et.al, 1996). According to Tall (1980), the idea of limit is first conceived as a process and then as a concept. The process of tending to a limit is a potential process that may never reach its limit (it may not even have an explicit finite procedure to carry out the limit process). This give rise to cognitive conflict in terms of concept images that conflict with the formal definition.

Concept Image and Concept Definition

Tall and Vinner (1981) used the term **concept image** to describe the total cognitive structure (in an individual's mind) that is associated with specific mathematical concepts while **concept definition** to be a form of words which specifies that concept. Such a concept definition may be formal and given to the individual as part of a mathematical theory. Concept image includes all the mental pictures and associated properties and processes. It is built up over time through all kinds of experiences and changes as the individuals meet new stimuli. Every individual has his/ her own unique concept image of a mathematical idea. So, an individual may invent concept definition to describe his/her concept image. Hence, the set of mathematical objects considered by the student to be examples of the concept is not necessarily the same as the set of mathematical objects determined by the definition. If these two sets are not the same, the student's behavior may differ from what the teacher expects. Cornu (1981) has examined the implications of this phenomenon for the learning process and pointed out that such a concept image turns into an obstacle when the student is faced with a situation where, because of its incompleteness, the concept image is insufficient. In other words, the student has a point of view that is too narrow, too exclusive, and thus inappropriate for dealing with a given situation or for solving a given problem.

Misconception

There are some disagreements among educators about the term 'misconception'. Some educators choose alternative understanding and others prefer to use un-matured concept. Gates (2001, p. 153) describe the dilemma when he said:

> I dislike the term as it appears to delineate a fixed boundary between ‘right’ and ‘wrong' ways of thinking. Some prefer the term ‘alternative mathematical framework’ to emphasis that the concept the pupil has is distinct from the culturally accepted one, yet is still reasoned and connected to other concepts. Misconception is not wrong thinking but is a concept in embryo or a logical generalization that the people have made. It may in fact be a natural stage of conceptual development.

According to Roseman (1985), the most flagrant and frequent mathematical errors made by remedial students in high school, emanates from failure to understand important concepts. He further explained that such misconception results in discouragement, absenteeism, and behavioral problems. Learning a mathematical idea, such as limit, involves a construction process. This implies that students build on and modify their existing concept image. The concept images, an individual’s constructs through his/her own activities, may differ in various respects to the formal mathematical concepts. This leads to the formation of alternative conception or misconception (Bezuidenhout, 1990). Educators attribute some of students’ failure in properly understanding mathematical concepts to their teachers. That is, teachers assume that either students have understood a mathematical concept at a high level of operational thinking when in reality they have a much lower level of mastery, or simplify a concept to the point of making it incorrect or misleading because of assuming that students would not understand if they are told all of the details (Asmamaw, 1993). Usually, serious misconceptions can arise when students are introduced to new mathematical meanings. Although there are different views on how to treat misconceptions, the sooner teacher recognize, diagnose, and address a misconception, the more effective he/she will be able to help students learning.

Conceptual verses Procedural Knowledge in mathematics

Conceptual and procedural knowledge in mathematics are topics addressed by many researchers. These two types of knowledge are assumed to be distinct yet related. Stump (2002) defined **conceptual knowledge** as knowledge that consists of rich relationships or webs of ideas. Schneider and Stren (2005) also described conceptual knowledge as knowledge of the core concepts and principles and their interrelations in a certain domain, knowledge that consists of those relationships constructed internally and connected to already existing ideas.

According to Engelbreeht et.al.(2007), conceptual knowledge is the ability to show understanding of mathematical concepts by being able to interpret and apply them correctly to a variety of situations as well as the ability to translate these concepts between verbal statements and their equivalent mathematical expressions. State of having conceptual knowledge is expressed by the ability to form connection between concepts or between concepts and procedures. Students use their conceptual knowledge to identify principles, what and when to use definitions and facts in mathematics and compare and contrast related concepts. Conceptual knowledge is knowing how or why to apply a concept that is adaptable, adjustable and applicable to other situation. Teaching conceptual understanding begins with posing problems that requires students to reason flexibly. Through the solutions process, students make connection to what they already know, thus allowing them to extend their prior procedural knowledge and transfer it to new situations (NTCM, 2002).

Procedural knowledge on the other hand is defined as knowledge of rules or procedures for solving mathematical problem, knowledge of operations and the conditions under which these can be used to reach certain goals (Schneider and Stern, 2005). It is the skill in carrying out procedures flexibly, accurately, effectively, and appropriately. It includes, but not limited to algorithms. That is, the step by step routines needed to perform arithmetic operations (The New York State Education Department as cited in Bosse and Bahr, 2005). It is the ability to solve a problem through the manipulation of mathematical skills, such as procedures, rules, formula, algorithms and symbols used in mathematics (Engelbreeht et.al.2007). Students use their procedural knowledge to solve a task in a step-by-step, sequentially ordered deterministic instructions. Procedural knowledge is to some degree automated, since its application, as compared to the application of conceptual knowledge, involves minimal conscious attention and few cognitive resources. Thus procedural knowledge is the knowledge of how to solve a problem through the manipulation of mathematical skills in a step by step sequential order. Much effort has been made examining the relation between conceptual and procedural knowledge in a mathematics class, particularly determining which takes the lead? To this end, Rittle-Johnson and Siegler (1998) concluded that there is no fixed order in the acquisition of mathematical skills verses concepts. In some cases, skills are acquired first; in other situation the order is reversed. Brown et.al (2002) argued for the importance of both conceptual and procedural knowledge and described the relation as: teaching first for conceptual knowledge leads to the acquisition of procedural knowledge, but the converse not to be true. Similarly, Schneider and Stren (2005) argued that conceptual knowledge is a source of children's procedural knowledge, but not vice versa.

Methodology

This study employed a descriptive survey method. Both qualitative as well as quantitative data were collected and analyzed. Descriptive survey is believed to be appropriate for the assessment of learning practices and procedures as well as learning difficulties and misconceptions. Koul (1996) noted that descriptive survey method helps to have general understanding of a problem by studying the current status, nature of the prevailing conditions, practice and trends through relevant and precise information. Moreover, Gay and Airasia (2000) indicate that descriptive survey is concerned with the assessment of attitudes, achievements, opinions, preferences, demographics, practices and procedures.

Population and Sampling Procedure

The population of this study was all students in government preparatory secondary schools of Dire Dawa city. During the period of this study there were three government preparatory secondary schools in Dire Dawa city *(Dire Dawa Comprehensive Secondary School, Sabian Secondary School, and New Secondary School)* which comprised a total of 479 grade 12 science stream students who were learning calculus. Literature suggests that sample size depends upon the nature of the population of interest and the data to be gathered and analyzed (Best et.al. 1993). However, Cohen and Manion (1994) have noted that a sample size of 25- 30 percent from the population is appropriate if the number of population is known. Accordingly, from the total of 479 grade 12 science stream students 135 (28 percent) of them were selected randomly to constitute the sample size for this study. The schools were used as strata so as to determine the sample that would be taken from each school. Hence, the overall sampling technique used for this study is a stratified simple random sampling where by the individual respondents from each school was taken by simple random sampling technique. The table below depicts the population and sample size used for the study.

Table 1: Population and sample size

Name of schools	No. of Students	No. of Sample students
Dire Dawa comprehensive secondary school	198	56
Sabian secondary school	149	42
New secondary school	132	37
Total	**479**	**135**

Data Collection Instrument

Achievement test was used as a major instrument for data collection. The aim of the test was twofold: the first is to examine how students conceive concepts in calculus, and the second is to identify misconceptions that they have developed. The test was evaluated and judged by panel of experts working on the area and accordingly the necessary improvements were made before the actual administration to the students. Furthermore, an item analysis was made for the test. To this end, the average difficulty level of the piloted test was found to be 48.7 percent and its average discrimination power was 0.32. The reliability of the test was also checked using Kuder Richardson formula 20 (KR-20). As a result, the reliability coefficient of the test was found to be 0.77 which is an acceptable level.

Results and Discussions

Students' Conception of Limit of Sequence

Students' test scripts were used as major data source to examine their difficulties, conceptions, and misconceptions. By way of examining conceptions and difficulties regarding limit of a function students were asked to respond to the following question:

Item 1: Choose the Correct Answer.

1.1. Given the sequence $\{3\}_{n=1}^{\infty}$ which one of the following do you think is correct about this sequence?

A. It is decreasing B. It is increasing

C. It is bounded D. All E. None of the above

Why do you think so? ______________________________

1.2. Which one of the following is true about a number sequence?

A .convergent sequence must be bounded. B. A bounded sequence must converge.

C. A divergent sequence must be unbounded. D. A monotone sequence must converge. Why do you think so? ______________________________

1.3 Which one is the limit of the sequence $\left\{1+\left(\frac{-1}{n}\right)^n\right\}_{n=1}^{\infty}$

A. 0 B. $\frac{3}{2}$ C. 1 D. $\frac{5}{4}$ E. has no limit

Give a reason for your answer ______________________________

1.4 If $\{a_n\}$ is a sequence, what criteria must be satisfied so that $\lim_{n\to\infty} a_n$ exist?

1.5. Let $a_n = \frac{1-\sqrt{n}}{2n+5}$, $n \geq 1$. then $\lim_{n\to\infty} a_n =$ ______________

(Write all the necessary steps and reasons which leads to the answer)

Table 2: Students' conception of limit of sequence

Question number	N=135											
	A		B		C		D		E		Non-respondent	
	f	%	f	%	f	%	f	%	f	%	f	%
1.1	9	6.7	41	30.3	26	19.2	29*	21.4	30	22.2	0	0
1.2	56*	41.4	16	11.8	30	22.2	18	13.5	10	7.4	5	3.7
1.3	16	11.8	8	5.9	89*	65.9	9	6.7	12	8.9	1	0.7

	Correct		Incorrect		Non-respondent		Total	
	f	%	f	%	f	%	f	%
1.4	36	26.7	79	58.5	20	14.8	135	100
1.5	61	45.1	61	45.1	13	9.6	135	100

* *Correct answer of the item*

Only few students (21.4 %) get the correct answer 'D' for question 1.1 while the large majority (78.6 %) of the students failed to get the correct answer of the question. Though the question is closed ended, students were asked to write justification and/or reason for their answer. Accordingly, the following were samples of reasons or justifications given by students:

Correct reasons:

S_{56}: *it is a constant sequence so it is bounded and satisfy property of increasing and decreasing sequence,*

S_{92}: *because constant sequence is both decreasing and increasing and is also bounded.*

Incorrect reasons:

S_{20}: $\{3\}_{n=1}^{\infty}$ *is a constant sequence therefore constant sequences are neither increasing nor decreasing but they can be bounded.*

S_{43}: *It is increasing, because as n goes to infinity, the function is increasing, not bounded, it is divergent, it does not have upper bound.*

S_{89}: *since the sequence given is a constant number it is neither increasing nor decreasing. And also for a function to be bounded or unbounded, it must be monotonic.*

On the basis of these reasons it can be asserted that students' wrong answers emanate from lack of knowledge about a constant sequence. Particularly students seem to have confusion between the common language and mathematical definition of terms. The terms like convergent, divergent, bounded and unbounded in calculus are possible sources for students' misconception. Difficulties to distinguish between convergence and boundedness, divergence and unboundedeness are most commonly observed problems among good numbers of students.

Only about 41.4% of students answered question 1.2 correctly while the remaining did not. From the given alternatives the one which reads '*a convergent sequence must be bounded'* is correct and all the rest are incorrect. Students were also asked to reason out their answer. Accordingly, many students reasoned out that ''*if a sequence is bounded then it has an upper bound and lower bound, which means it has maximum and minimum , If so it converges to one of these extreme values''.* But this is true only for a monotone sequence. The fact that there are few number of students who wrote the correct reason compared to those who get the correct answer indicates that getting correct answer by itself does not necessarily indicate students' correct conception and understanding of the concept.

Interesting results were obtained from question 1.3 of the test where 65.9% of the students answered correctly. Although significant number of students responded correctly to this question, good numbers of them were unable to give correct reason for their correct answer. For instance, some of the incorrect reasons read as: ''*because when we insert* ∞ *in the value of n the result is one i.e.* $1+\left(\frac{-1}{\infty}\right)^{\infty}$ *=1.''* It is therefore apparent that those students having such reasons for their answer conceive infinity as a number and limit as just substitution.

The purpose of question 1.4 of the test was to examine whether or not students were able to determine the criteria for existence of limit of a sequence beyond computing limit of a specific sequence. Similarly, students' response to question 1.5 of the test tells about their knowledge of process of solving a non- routine exercise on limit of a sequence. The correct answer to question 1.4 is: **as n tends to infinity a_n must converge to a unique number.** However, only 26.7 % of students responded correctly to this question. The following students' excerpt , for instance, are among the incorrect ones: a) the two side limit (i.e. right and left hand limit) must be equal), *b)* $\lim_{n\to\infty^-} a_n = \lim_{n\to\infty^+} a_n = L$ .It seems that students failed to understand that at ∞ there is no approach to both sides which is also true for functions. To this end, both responses indicate that students hold confusion between limit of a sequence and limit of a function. Limit of sequence involve only moving in one direction whereas limit of a function involve moving both to right and left of a point. It is also observed that some students conclude every monotonic sequence is bounded. Some argued that monotonic is sufficient condition for convergence.

Equal proportions of students (45.1%) answered question 1.5 correctly as well as wrongly. The correct answer is zero. From this specific question the following types of arguments that might have led towards misconception have been observed from students' test excerpts: simplifying $\frac{-\sqrt{n}}{n}$ as 1, over generalization of properties of limits of functions like equating $\lim_{n\to\infty} \frac{1-\sqrt{n}}{2n+5}$ with $\frac{\lim_{n\to\infty}(1-\sqrt{n})}{\lim_{n\to\infty}(2n+5)}$, $\frac{-1}{\infty} = 0$ (which emanates from considering ∞ as a number).

To this end, the views that a constant sequence is neither increasing nor decreasing, every sequence can be categorized as arithmetic or geometric, considering ∞ as a number, and over generalization of properties of limits of functions are among difficulties students encountered in understanding limit of a sequence.

Students' conception of limit of functions

The purpose of item 2 of the test was to examine how students understand limit of functions and whether they conceive limit of a function at a point as reachable or unreachable and/or as an approximation or not.

Item 2: Mark the following statements as true or false

2.1 A limit is a number in which the function value cannot exceed.
2.2 A limit is a number that the function value gets closer to but never reaches.
2.3 A limit is only an approximation that can be made as accurate as you wish.
2.4 The limit of a function can fail to exist at a certain point.

As indicated inTable-3a below most students hold incomplete conception of the concept of limit of a function.

Table 3 a: Students' conception of limit of functions

Item number	N=135						Total	
	True		False		Non-respondent			
	f	%	f	%	f	%	f	%
2.1	57	42.2	75*	55.6	3	2.2	135	100
2.2	94	69.6	39	28.9*	2	1.5	135	100
2.3	45	33.3	88*	65.2	2	1.5	135	100
2.4	102*	75.6	32	23.7	1	0.7	135	100

** Correct answer*

Questions 2.1, 2.3 and 2.4 were correctly answered by the majority of students .However, when the proportion of students who answered these questions wrongly together with the large proportion of students who failed to answer question number 2.2 are considered, it indicates the presence of misconception on the part of some students .With regard to question 2.2, about 28.9% of the students responded as **false** to the statement *'a limit is a* *number that the function value gets closer to but never reaches'*. Thus, these students conceive limit of a function as reachable. Whereas 69.6% of them agreed with the statement *'a limit is a number that the function value gets closer to but never reaches'* is **true**. Although this information may not be adequate to conclude whether or not students have clear knowledge of reachable or un-reachable of limit value at least it indicates that they have the conception that limit values are not reachable. Often students' conception of limit value arises from working with functions which are defined at the limit points (like polynomial functions). Besides, 65.1% of the students correctly answered that a limit is not only an approximation that can be made as accurate as we wish. Only 33.3% of them have the opinion that limit is just an approximation as we wish.

Question 2.4 has a high correct response (75.6%). But still there are 23.7% of students who expressed their disagreement to the statement that '*the limit of a function can fail to exist at a certain point'*. This has an implication that these students may have a concept image which may not agree with the formal definition since in the formal definition what makes function value different from limit value is clearly stated (i.e. limit value may exist at a point which is not in the domain of the function). Thus, from the aforementioned data, it can logically and safely be asserted that there are students who conceive limit of a function as a maximum value at points near the limit point, unreachable, an approximation that can be made as accurate as we wish and a value which exist at any point. Such conceptions of limit as unreachable, maximum value, and approximation are virtually misconception as they occur not due to lack of knowledge rather due to un-matured concept. By way of substantiating the aforementioned result about students' conception of limit, students were also asked to respond to the following open-ended question:

> ***Item 3****:* Given a function f and a number C. Describe in your own words what it means to say that the limit of a function f as $x \to C$ is some number L?

Table 3 b: Students' conception of limit of functions

Item number	N=135						Total	
	Correct		Incorrect		Non-respondent			
	f	%	f	%	f	%	f	%
3	26	19.3	96	71.1	13	9.6	135	100

The purpose of item number 3 was to assess students' own concept definition of limit of a function at a point. Accordingly, 9.6% of the students failed to give any answer while only 19.3% of them described correctly. Some of the correct answers were:

> ***S$_{62}$****: As x approaches to 'c' from the right and left if f has the same result 'L'.*
>
> $$i.e.\ \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$$
>
> ***S$_{108}$****: When x approaches c but does not mean x=c, if the value of f is the same*
>
> *number L.*
>
> ***S$_{10}$****: The value f gets closer and closer to a number L as the value of x approaches*
>
> *to c but not necessarily equal to c.*

However, the majority of students (71.1%) were unable to correctly describe the answer. The incorrect responses forwarded by students include but not limited to: $\lim_{x \to c} f(x) = L$. Apparently such student has a confusion between symbol and meaning, as $\lim_{x \to c} f(x) = L$ is nothing but a symbol used to abbreviate the phrase '*the limit of **f(x)** as **x** approaches to **a** is **L'***. Some others preferred to compute the limit of specific functions to describe their concept image. For instance, the following are excerpts from students' test papers:

S$_7$: $\lim_{x \to 2}\left(x^2 + 2x + 3\right) = 11$, S$_{22}$: $\lim_{x \to 0} x^2 + x + 2 = 2$, S$_{97}$: $\lim_{x \to 0} \frac{3x+5}{4x+5} = \frac{3}{4}$

This group of students conceive limit exclusively as substitution and they don't differentiate limit value from function value. Similarly there were students who responded to item number 3 as follows: S71: $\lim_{x\to 1}\frac{x+1}{x^2+2x+1}=\lim_{x\to 1}\frac{x+1}{(x+1)(x+1)}=\frac{1}{2}$, S45: $\lim_{x\to 0}\frac{x^2-1}{x-1}=\lim_{x\to 0}\frac{(x-1)(x+1)}{x-1}=2$

This group of students have the conception that limit is an alternative way to calculate value of a function when only the function is not defined at the limit point. But we can speak about limit of a function at a point whether the function is defined or not. The result of item 3 revealed that most students lack matured concept image of the concept of a limit to be described in words. This is not a mere language problem, rather un-matured level of conceptual knowledge (concept image).

Items 4, 5 and 6 of the test were intended at examining the presence or absence of misconceptions among students regarding two sided limit, vertical asymptote, limit at a point, and absence of a limit at a point.

Item 4: Let f be a function and c ∈ R. If $Lim_{x\to c} f(x)$ does not exist, which one is necessarily true?

A. $Lim_{x\to c^+} f(x)$ exist but different from $Lim_{x\to c^-} f(x)$

B. f(x) becomes large enough when x gets closer and closer to c.

C. The function has a vertical asymptote at x=c.

D. f(x) is defined at x=c

E. None

Please give a reason for your answer ______________________________

Item 5: Given a function f such that $Lim_{x\to 2} f(x) = 3$ .Which of the following statements must be true about the function f?

A. f(2)=3 C. f is continuous at the point x=2

B. f (x) is defined at x=2 D. $Lim_{x\to 2^+} f(x) = 3$ E. None

Why do you think so? ____

Item 6: Evaluate the limit of the following functions (Show all the necessary steps clearly)

6.1. f(x) = $\frac{\sqrt{x+1}+1}{x}$ at x =1 6.2. g(x) = $\frac{\sqrt{x^2+9}-3}{x^2}$ at x=0

Table 4: Students' response to items 4, 5 and 6

Item number	N=135											
	A		B		C		D		E		Non-respondent	
	f	%	f	%	f	%	f	%	f	%	f	%
4	76	56.3	9	6.7	23	17.0	11	8.1	13*	9.6	3	2.2
5	41	30.3	17	12.6	20	14.8	38*	28.1	10	7.4	9	6.7
6	Correct				Incorrect							
	f		%		f		%				f	%
6.1	87		64.5		29		21.5				19	14.0
6.2	60		44.5		56		41.5				19	14.0

** Correct answer of the item*

The data in table 3 portrays that only 9.6% of students answered item 4 correctly. These students comprehend that limit of a function fail to exist for different reasons. About 56.3% of students who decided 'A' as their answer seem to have the opinion that the only case where limit of a function fail to exist is that when different one sided limits are obtained. The following are samples of students' reasons to their answers:

$S_{76:}$ because the function left side is negative and the right side is positive.

This student misconceived that $x \to c^{+}$ (when x tends to c from the right) as if c is positive and $x \to c^{-}$ (when x tends to c from the left) as if c is negative.

S_{46}: if $\lim_{x \to c} f(x)$ does not exist then the function has a vertical asymptote at x=c.

This student ignored the non existence of a limit because of different left and right limit.

Table 3 also indicates that only 28.1% of the students answered item number 5 correctly while 30.3% of the students have chosen 'A' as a result of their opinion that limit of a function is obtained by substitution. Besides, 14.8% of them hold the opinion that a function must be continuous to have a limit at a point although the reveres holds true (i.e. if a function is continuous then it must have a limit). About 12.6% of students have the opinion that a function must be defined to have a limit at a point. Confusing right and left hand limits, the opinion that limit value is always obtained by substitution, the believe that if a function has a limit at a point then it is continuous are the observed over generalizations that students hold. About 64.5% and 44.5 % of students have answered correctly questions 6.1 and 6.2 respectively. However, there is a big difference between number of students who answered both questions correctly and between those who answered only one and missed the other question. The disparity in the number of respondents for questions 6.1 and 6.2 could be explained by the same fact that was observed previously when students saw limit at a point as substitution. Additionally it was observed, a generalization from students' computation, that rationalization seems a must to do when radical is involved in a limit.

Confusion about two sided limit, the belief that if a function has no limit at a point then it must have a vertical asymptote, if a function has a limit at a point then it must be defined and continuous at that point are some of the observed difficulties and misconceptions students formed in learning concepts of calculus.

Students' Conception of Continuity

Students' lack of skills to sketch graphs of functions is one of the factors affecting their understanding of the concept of continuity. Item 7 of the test was aimed at examining whether students are able to sketch graph of a function. Moreover, it was aimed at examining their conception of a function undefined at a point, existence of a limit at that point, and implications to discontinuity. Furthermore, in order to assess students' ability and conceptions about the relationship between limit and continuity, right and left hand limit with continuity, they were asked to respond to item 8 of the test.

Item 7:

7.1 Sketch the graph of the function f(x) = $\frac{x^2-9}{3x-9}$ and answer the following questions.

7.2 What happens to the graph of f at the point x=3?

7.3 Is the function (continuous/discontinuous) at the pint x=3?

7.4 What is the limit of f at x=3?

7.5 What is the value of the function at x=3, i.e. f (3)?

Item 8: Find a and b that will make the function f continuous in $(-\infty,\infty)$ if

$$f(x)=\begin{cases}3x+1,.........x<2\\ ax+b,.......2\le x<5\\ x^2,.............x\ge 5\end{cases}$$

Table 5: Students' responses to items 7 and 8

Item number	N=135						Total	
	Correct		Incorrect		Non-respondent			
	f	%	f	%	f	%	f	%
7.1	25	18.5	51	37.8	59	43.7	135	100
7.2	48	35.6	80	59.2	7	5.1	135	100
7.3	81	60	48	35.6	7	4.4	135	100
7.4	80	59.2	51	37.6	4	2.9	135	100
7.5	87	64.5	41	30.3	7	5.1	135	100
8	47	34.8	39	28.9	49	36.2	135	100

As can be seen from Table-5, significant number of students (43.7%) attempted to answer questions 7.2 - 7.5 without sketching the graph of the function $f(x)=\frac{x^2-9}{3x-3}$. Although a very good way to analyze limits is to look at the graph of the function in a neighborhood of the limit point x_0 (Stump, 2002), a good number of students failed to do so. While only few students (18.55%) were able to sketch the graph and used it to answer the subsequent questions about 37.8% of them sketched it wrongly. The figures below depict samples of students' sketches of the graph:

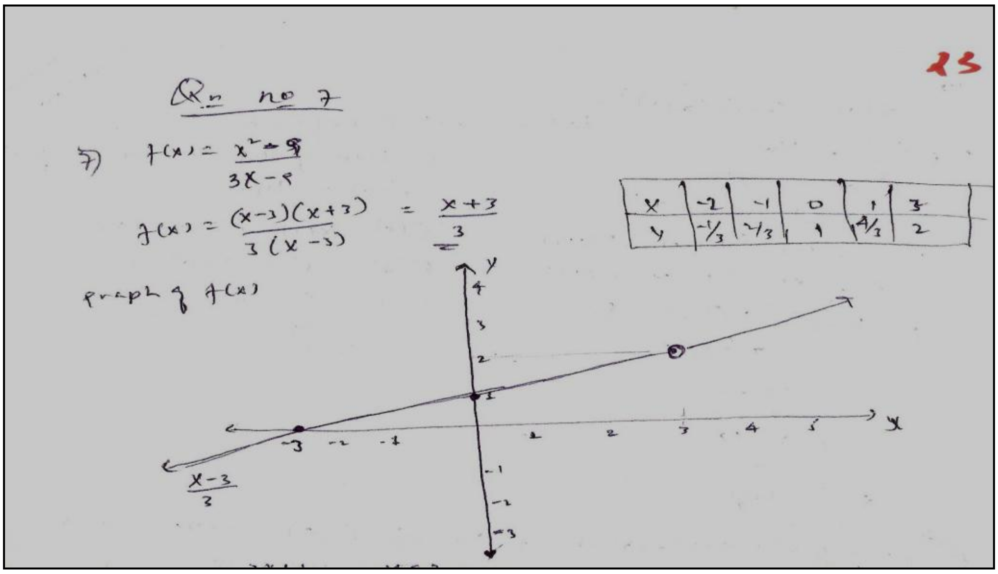

Figure 1: Students' correct sample graph

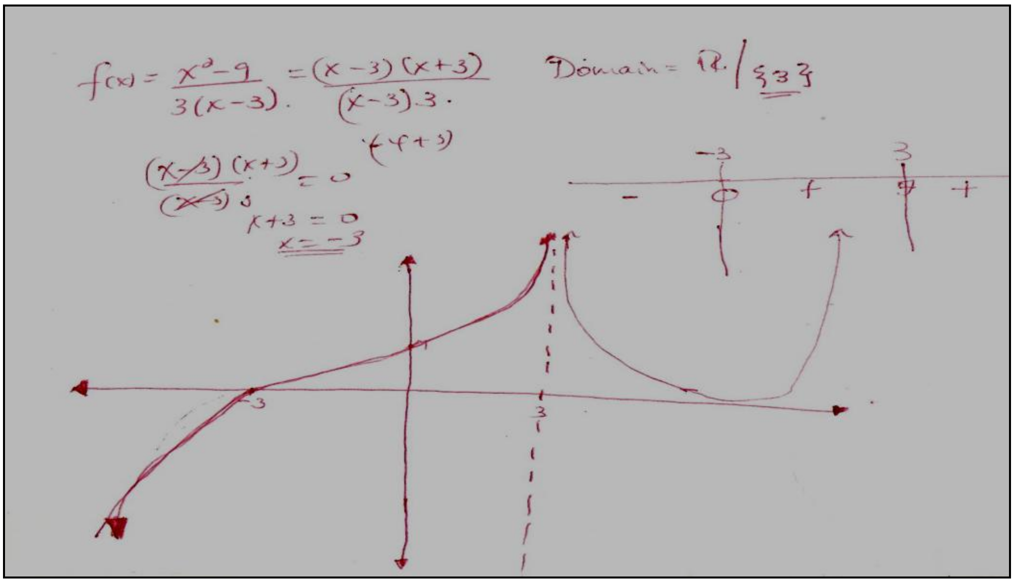

Figure 2: Students' incorrect sample graph

Disregarding restriction to the domain when joining points to sketch a graph, drawing the graph without considering sufficient points which lie on the graph, and considering every point of discontinuity as an asymptote are some of the possible factors which led students sketch incorrect graph for the given function. When students were asked to '*what happens to the graph of f at x=3*' about 35.6% of them responded correctly that **the graph makes a hole**. On the other hand, the following were some of the incorrect responses given by students*: it has a vertical asymptote, it has an asymptote, zero, there is a graph, f(x) is increasing, the value becomes 2 and continuous, the graph will be continuous, and the graph is at infinity, it decreases .*

Students' responses to questions 7.1 and 7.2 are good indicators of their knowledge of a function as *'full of symbolic notation'*. Their responses also indicate how they understood points of discontinuity as an asymptote. When students were asked whether or not the function is continuous/discontinuous at the point x= 3, about 35.6% of students responded that the graph is continuous at the point x=3. Similarly, when students were asked the value of the function at x=3, *i.e. f (3) ,* the majority of students (59.2%) answered it correctly while few of them (37.8%) responded incorrectly. Some students whose responses were wrong failed to examine the graph and instead tried to answer the subsequent questions by substitution. This means they conceived limit as just substitution indicating a presence of misconception. Some others responded that '*the limit does not exist'* which still indicate their misconceptions. The following are samples of students' arguments for their assertion which led them to misconception:

- Since $\lim_{x\to 3^-}\frac{x+3}{3}=\frac{-3+3}{3}=\frac{0}{3}=0$ and $\lim_{x\to 3+}\frac{x+3}{3}=\frac{3+3}{3}=2$ then the limit does not exist.
- $\frac{x^2-9}{3x-9}=\frac{3^2-9}{3*3-9}=\frac{0}{0}=\not\in$ (does not exist)

❖ The limit is zero. Some of their procedures are the following:

$\lim_{x\to 3}\frac{x^2-9}{3x-9}=\frac{-9-9}{9-9}=\frac{0}{0}=0$, $\lim_{x\to 3}\frac{x^2-9}{3x-9}=\lim_{x\to 3}\frac{(x-3)(x+3)}{3(x-3)}=\lim_{x\to 3}\frac{x+3}{3}$

then at x=0, $\frac{0+3}{3}=0$

- One student said that limit is the same as y-intercept and he proceeded as follows $y=\frac{x^2-9}{3x-9}=\frac{(x+3)(x-3)}{3(x-3)}=\frac{x+3}{3}$. At x=0, $y=\frac{0+3}{3}=1$.

The purpose of item 8 was to assess whether or not students could relate concepts of limit and continuity. Students need to apply the concept of one sided limit at a point to answer this question. Accordingly, 36.3% of the students declined to respond. However, 34.8% of them answered it correctly by demonstrating all the necessary steps clearly. The rest 28.9% applied the necessary concept but failed to get the correct answer because of simple arithmetic they overlooked such as the following:

S_{6}: $a = \frac{7-b}{2}$ and $5a + b = 25$ then $5\left(\frac{7-b}{2}\right) = 25$,

S_{23} and S_{49}: $\left.\begin{matrix} 5a + b = 25 \\ 2a + b = 7 \end{matrix}\right\} \Rightarrow 3a = 18 \Rightarrow a = 3$, S_{19}:

$\left.\begin{matrix} -2a - b = 7 \\ 5a + b = 25 \end{matrix}\right\} \Rightarrow 3a = 8 \} \Rightarrow a = \frac{8}{3}$

In addition, 5.1% students misused the concept of continuity. This is apparent when they demonstrated their understanding as follows:

S_{36} and S_{59}: $\lim_{x \to 2^-} ax + b = \lim_{x \to 5^+} x^2$ $\quad \begin{cases} 2a + b = 25 \\ 2a + b = 7 \end{cases} \Rightarrow 0 = 8 \rightarrow\leftarrow$

S_{103}: $\lim_{x \to 2^-} f(x) = \lim_{x \to 5^+} f(x) = \lim_{x \to 2} f(x) \quad \Rightarrow \quad ax + b = 3x + 1,$

$\Rightarrow \quad a(x-3) = 0$ Or $b - 1 = 0 \quad \Rightarrow \quad a = 0$, $x = 3$ and $b = 1$

Some students missed not only the concept but also have shown serious algebraic manipulation problems. Thus, students' conception of continuity is influenced by their knowledge of graph, algebraic manipulation and the concept of asymptote.

Students' Conception of Derivative

The purpose of items 9 to 12 was to examine students' difficulties and misconception regarding the concepts of derivative. Students were asked to respond to the following questions:

Item 9: given a function **f** and a number **a** in the domain of **f**. consider the expression

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$, provided the limit exists.

9.1 What symbol do we use to represent this quantity?
9.2 What is the name of the symbol we use to represent this quantity?
9.3 What is the meaning of this quantity?

Item 10: Let $g(x)=(x-1)(x+2)^2$ then

10.1 Compute $g'\left(\frac{1}{2}\right)$

10.2 Find slope of a line tangent to g at (1, 0)

Item 11: Let f(x) = $\frac{1}{3}x^3-\frac{9}{2}x^2+18x-5$ then

11.1 Compute $f'(1)$

11.2 Find extreme value of f on $[2,4]$ and determine at which number it occur

Item 12: Find **a** and **b** such that **f** is differentiable at x=1 if

$$f(x)=\begin{cases} ax,....................x\le 1 \\ bx^2+x+1,........x>1 \end{cases}$$

(Show all the necessary steps clearly)

Item 9 intends to check whether students could identify symbol, name, and meaning of an expression used to introduce a new concept. Accordingly, it is observed that they confuse symbol with name and meaning of an expression which in turn affected their understanding. Derivative could be introduced as a limit of the difference quotient of a given function or as an instantaneous velocity (as limit of average velocity). In grade 12 Mathematics text book, derivative is introduced using the first approach. Accordingly, the meaning of derivative is slope of a line tangent to graph of a function at a point. The symbols used to represent this quantity are: m_a, f'(a), $\frac{df}{dx}(a)$

Table 6: Students' response to item number 9 and 10

Item number	N=135						Total	
	Correct		Incorrect		Non-respondents			
	f	%	f	%	f	%	f	%
9.1	92	68.1	31	22.9	12	8.9	135	100
9.2	75	55.6	49	36.2	11	8.1	135	100
9.3	63	46.7	60	44.4	12	8.9	135	100
10.1	72	53.3	49	36.2	14	10.3	135	100
10.2	68	50.3	51	37.8	16	11.8	135	100

About 68% of the students answered question 9.1 correctly. The following are some of the correct as well as incorrect responses to the question: $f^1(a)$, m_a, $f^1(x)$, $\frac{df}{dx}$, *derivative, slope, slope of a tangent line*. From this list only the first two are correct. Derivative is not a symbol rather a name of a quantity obtained from the expression '$\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$'. When students were asked (question 9.2) to write the name of the symbol we use to represent the quantity '$\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$' the majority of them (55.6%) answered it correctly as '**derivative of *f* at x=a**'. Similarly, students were asked about the meaning of the quantity '$\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$' provided the limit exists and only 46.7% of them answered it correctly. The following were some of their responses: '*the derivative of f(x), slope of a line, tangent of a line, the function is differentiable at x=a, the value of a function as x goes to a, the inclination of the function f '.* Thus, students' responses to questions 9.1 - 9.3 show that there is no clear identification of symbol, name of a symbol and meaning of a quantity obtained from an expression. To compute derivative of a function at a point students have three approaches: to apply derivative of a combination and composition function, expanding the given expression and then applying properties of a combination of function, or to apply the definition of derivative at a point (limit of the difference quotient). To this end, 53.3% of the students arrived at the correct answer $\frac{15}{4}$ to question 10.1, while 36.2% of them have arrived at an incorrect result and the remaining 10.4% failed to give any response. Out of 40% students who used the first approach 30.3% of them got correct and out of 12.6% who have used the second approach 9.6% of them succeed. No student used the third approach for this question. The following are sample of students' responses that used correct procedure to solve question 10.1 using the first and second approaches mentioned above respectively:

S_{48}: $g(x)=(x-1)(x+2)^2$ let A=(x-1) and B=$(x+2)^2$ then A'=1 and B'= 2(x+2)
But , $g'(x)=A'B+AB' \Rightarrow g'(x)=(x+2)+2(x+2)=3(x+2)$.

This student have a clearly image about derivative of a composition and combination function.

S_{64}: $g(x)=(x-1)(x+2)^2$ then $g(x)=x^3+4x^2+4x-x^2-4x-4=x^3+3x^2-4$
$\Rightarrow g'(x)=3x^2+6x$

While an example of incorrect solution is:

S_4: $g(x)=(x-1)(x+2)^2=(x-1)(x+2)(x+2)$ then let $a=(x-1)$, $b=(x+2)^2$
$g'(x)=a'b+ab' \Rightarrow g'(x)=(x+2)^2+(x+1)(2x+4)=3x^2+10x+8$

Similarly, 50.3% of the students have got the correct answer of question 10.2, while 37.8% of them were unable to get the correct answer. The fact that question 10.2 is deliberately trivial and that good numbers of students either failed to get the correct answer or failed to respond at all indicates absence of conceptual understanding on the part of students. That is lack of conceptual knowledge that derivative of a function is again a function evaluated at a point which gives a quantity, slope of a line tangent to the given function at that point.

We also observed that students consider derivative of a function which is obtained using the formula (the limit of the difference-quotient) different from the one obtained using techniques of derivatives. Most of these students might think that the slope of a tangent to graph of a function is the one which is obtained from limit of the difference quotient. A student who sees these two limits as different presumably lacks breadth of understanding. Some students start the solution correctly and end with incorrect result perhaps due to problem of algebraic manipulation, some students are confusing slope and equation of a tangent line. Generally students' response to item number 10 indicates their difficulties of conceptual understanding.

The purpose of items 11 and 12 were to students to assess students' understanding of derivative in deferent forms and their skills and difficulties in finding extreme value.

Table 7: Students' response to Item numbers 11 and 12

Item number	N=135						Total	
	Correct		Incorrect		Non-respondents			
	f	%	f	%	f	%	f	%
11.1	79	58.5	38	28.1	18	13.3	135	100
11.2	31	22.9	84	62.2	20	14.8	135	100
12	21	15.6	65	48.1	49	36.2	135	100

Question 11.1 asks students to compute derivative of a polynomial function. The majority of students (58.5%) arrived at the correct answer, while only 22.9% of them responded correctly to question 11.2 that demand a connection between concepts to find extreme value in the given interval. Besides, it has been observed from students' test scripts that students have difficulties in understanding computation of derivatives of a function at a given point and extreme values. More specifically good numbers of students are equating critical number with extreme value, have the view that extreme value occurs only at the boundary points, and that extreme value occurs only at the critical points.

By way of assessing the extent to which students form connection among concepts to solve a problem in general and their ability or difficulty of creating a connection among the concepts of limit, continuity and derivative in particular, they were asked to respond to item 12 of the test. The solution of this item demands an integrated approach of limit, continuity and derivatives. The fact that f is differentiable at x=1 tells us two things; f is continuous at x=1, and that the limit difference-quotient exists. To this end, significant number of students (36.2%) failed to give any response to the question. Among those (63.7%) who have attempted the same question, only 15.6% of them have responded correctly. The following were sample of students' responses showing their correct as well as incorrect procedures in solving item 12:

Correct procedure:

S_{63}: $f(x)=\begin{cases} ax,....................x \le 1 \\ bx^2+x+1,........x>1 \end{cases}$

$\lim_{x\to 1} f(x) = \lim_{x\to 1^=} f(x) = \lim_{x\to 1^+} f(x)$ So, $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(bx^2+x+1) = b+2$ and

$\lim_{x\to 1^=} f(x) = \lim_{x\to 1^-} ax = a$

$\Rightarrow b+2=a$

$$\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} \Rightarrow \lim_{x\to 1^-}\frac{(ax)-a}{x-1} = \lim_{x\to 1^+}\frac{(bx^2+x+1)-a}{x-1}$$

$$\Rightarrow a = \lim_{x\to 1^+}\frac{bx^2+x+1-b-2}{x-1} = 2b+1$$

So, b=1 and a=3

Incorrect procedure:

S_{75}: $f(x)=\begin{cases} ax,....................x \le 1 \\ bx^2+x+1,........x>1 \end{cases}$ $\Rightarrow \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) \Rightarrow a = b+2$

$f(x) = \lim_{x\to 1} f(x) \Rightarrow ax = bx^2+x+1$??

S_{107}: $f(x)=\begin{cases} ax,....................x \le 1 \\ bx^2+x+1,........x>1 \end{cases}$ $\Rightarrow \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1)$

$\Rightarrow ax = bx^2+x+1 = ax \quad \Rightarrow a = b+2+a$

$\therefore a = a$ and $b+2=a$

It has been observed from students' test script that about 28% of them wrote $\lim_{x \to 1} f(x) = f(1)$. These students understood that a differentiable function is continuous. On the other hand, about 32.5% students wrote $\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$ exist. These students have the concept that a function is differentiable and the limit of the different quotient exists. But only 25.7% students split this concept in to equality of the right and left hand limit .Only 20.7% of students arrived at the correct second equation (2b+ 1=a) and then only 15.6% of them arrived at the final correct answer. Few students (2.2%) were even unable to remember method of solving system of equations simultaneously. Good numbers of students also misconceive function of the type $f(x) = \begin{cases} ax, \ldots\ldots\ldots\ldots\ldots\ldots x \le 1 \\ bx^2 + x + 1, \ldots\ldots\ldots x > 1 \end{cases}$ as two distinct functions. As a result, students were unable to relate right and left hand limit.

Conclusion

The current finding is not far from what literature says about difficulties in learning calculus. Cornu (1981), Schwarzenberger & Tall (1978) , and Orton (1980ab), for instance, have identified the following fundamental difficulties with limit and infinite process:

- difficulties embodied in the language; terms like "limit'', " tends to'', "approaches'', "as small as we please'' have powerful colloquial meanings that conflict with the formal concepts;
- the limit process is not performed by simple arithmetic or algebra, infinite concepts arise and the whole thing becomes "surrounded in mystery'';
- the process of "a variable getting arbitrarily small'' is often interpreted as an "arbitrarily small variable quantity'', implicitly suggesting infinitesimal concept even when these are not explicitly taught ,
- likewise , the idea of "N getting arbitrarily large'' , implicitly suggests conception of infinite numbers;

- students often have difficulties over whether the limit can actually be reached; and
- there is confusion over the passage from finite to infinite, in understanding "what happens at infinity''.

Furthermore, restricted mental images of functions, difficulties in translating real-world problems in to calculus formulation, difficulties in selecting and using appropriate representations, difficulties in absorbing complex new ideas in a limited time, and consequent student preference for procedural method rather than conceptual understanding are among the factors mentioned in the existing literature that have contributed for students' difficulties and misconceptions in learning calculus.

To this end, students' pre-calculus knowledge is apparently more procedural than conceptual. Significant numbers of students in secondary schools of Dire Dawa city are observed to have difficulties and formed misconceptions on basic concepts of calculus. More specifically, they conceive that:

- a constant sequence is neither increasing nor decreasing;
- a monotonic sequence never bound;
- an infinite sequence is never bounded;
- all sequences can be categorize as arithmetic or geometric;
- limit as a number in which the function value cannot exceed;
- limit of a function at a point as unreachable;
- limit at a point as a function value at the limit point; and
- every point of discontinuity as a vertical asymptote.

Good numbers of students possess only a limited concept image of the concept of limit of function. Their concept definition of a 'limit' is incompatible with the formal concept definition of limit. Students preferred to describe a limit using a counter function instead of writing a formal definition. Good numbers of students lack the necessary knowledge and skills of representing function using different methods. There are also students with restricted mental image of functions to the extent that they lack knowledge of algebraic manipulation.

Although the majority of students can compute derivate of a function good numbers of these students, however, were found to have difficulties in finding slope and extreme value(s) of function. Put in other words, it seems that students can perform procedural aspects of the mathematics better than the conceptual aspects. Students are also observed to have difficulties and lack of conceptual understanding to generate meaning to the calculated derivative value. There are students who assumed that a constant sequence is not monotonic. This could be due to their difficulties with meaning of the word 'increasing/decreasing' in the common language. Some students conceptualize that a monotone sequence is never bounded while the majority of students conceive that any sequence can be categorized as either arithmetic or geometric. That is, students conceive that every sequence have an obvious consistent pattern. There are also students with strong understanding that limit of a function at a point is unreachable. This group of students conceive limit as a dynamic process involving infinitely many steps. Others assumed that limit is just an approximation that can be made as accurate as we wish. Some others still equate limit at a point as equal to the function value at the limit point. That is, students consider limit values are always computed just by substitution. This might have been developed from exercising many cases of functions which are defined at the limit point. Although most students are aware that limit of a function may fail to exist at a point, they interpret this wrongly as there must be an asymptote to the graph of the function at that point. On the other hand, most students have difficulties in constructing a statement that describes their concept image of a limit. Instead of describing the concept of limit in terms of words, majority of students preferred to show it by computing limit of a specific function.

The fact that students have encountered difficulties of algebraic manipulation, restricted mental image of function, applying the correct concept, applying chain of concepts to solve a problem indicates their lack of depth understanding of the basic concepts of calculus. This is apparent, for instance, when significant numbers of students were able to compute derivative of a function but failed to relate with its application. There are students who consider derivative of a function obtained from the difference-quotient to be different from the one obtained by applying techniques of derivatives. Students have not only over generalized properties of limits but also have developed misconception that they perceive; the limit of a function at a point is always equal to the function value at the point; a function must be defined at a point to have a limit at that point. On the bases of students' test results and series of classroom observations we have made, it can logically and safely be asserted that the teaching-learning process in mathematics classrooms as well as the practice of assessment were dominantly procedural. By and large, inadequate and poor level of understanding of pre-calculus concepts, lack of commitment on the part of students , traditional beliefs that calculus is difficult ,and teachers' pedagogical limitation in teaching concepts , are presumably factors which negatively influenced students' conceptual understanding of concepts of calculus .

Hence, we recommend that mathematics teachers teaching concepts of calculus for beginners (secondary level) must take the necessary pedagogical-content care. They are also advised to provide highly vivid illustrations by way of maximizing students' understanding and avoiding possibilities of misconception on the parts of their students. Teachers are also advised to use visual demonstration in teaching calculus so that students could grasp the practical ideas of calculus. We also recommend that local educational authority need to assist the teaching learning process by way of providing need-based training to mathematics teachers. Furthermore, the federal ministries of education should revisit persevere as well as in-service teacher education programs and gear the training towards pedagogical content knowledge.

References

Asmamaw Yimer. (1993). *The Effectiveness of Teaching Mathematical Concepts by the Inductive and Deductive Approaches: An Experiments study in Grade Eleven of Entoto Comprehensive Secondary School.* Master's Thesis, Addis Ababa University (unpublished).

Best J. W. and Kahn, J. V. (1993). **Research in Education** (7th ed). New Delhi: prentice-Hall of India Pvt. Ltd.

Bosse, M. J. and Bahr, D. L. (2005). *The State of Balance between Procedural Knowledge and Conceptual Understanding in Mathematics Teachers Education.* Retrieved from http://www.Cimt.plymouth.ac.uk/journal/bossebahr.pdf

Cohen, L. and Manion, L. (1994). **Research Method in Education** (4th ed). Routledge New York.

Cornu, B. (1981). *Advanced Mathematical Thinking*. In Nesher, P. and Kilpatrick, J. (eds), **Mathematics and Cognition**.(pp 113-134).

Cottril, J. et.al. (1996). *Understanding the Limit Concept: Beginning with a Coordinated Process Schema.* **Journal of Mathematical Behavior**, 15,167-192.

Engelbreeht, J. et.al, (2007). *Understanding Students' Performance and Confidence in Procedural and Conceptual Mathematics.* Retrieved from http://Science.up.ac.za/muti/conceptualmath.pdf.

Ellis, R. and Gulick, D. (1993). **Calculus with Analytic Geometry**. (5th ed).

Gay,L.R. and Airasia,P.(2000). **Educational Research: Competencies for Analysis and Application** (6th ed). New Jersey: prentice-Hall, inc.

Koul, L. (1996). **Methodology of Educational Research**. School Revised Edition. New Delhi: Vikas.

Larcombe A. (1985). **Mathematical Learning Difficulties in the Secondary School**. Philadelphia: Open University Press.

MoE. (2006$_a$). **Mathematics Syllabus for Grade 11 and 12**. Ministry of Education, Addis Ababa, Ethiopia

MoE.(2006$_b$). **Mathematics students' textbook, Grade 12**, *Revised Edition*. Ministry of Education, Addis Ababa, Ethiopia

NTCM. (2002). **Curriculum and Evaluation Standard for School Mathematics**. Reston,VA: Author.

Orton,A.(1980a). *A cross-sectional study of the understanding of elementary calculus in adolescents and young adults*, unpublished PhD. Leeds University.

Orton, A.(1980b). *An investigation into understanding of elementary calculus in adolescents and adults, Cognitive Development Research in Science and Mathematics*, University of Leeds 201-215.

Rittle-Johnson, B. and Siegler, R.S. (1998). *The relation between conceptual and procedural knowledge in learning mathematics: A review.* In Donald, C (ed.). **The development of mathematical skills** (pp. 75-328). Hove, UK: Psychology Press.

Roseman, Louis (1985). *The Essential Concepts for Remediation in Mathematics*. **The Mathematics Teacher**, vol. 78, No. 7

Schneider, M. and Stren, E. (2005). *Conceptual and Procedural Knowledge of Mathematics Problem: Their Measurement and Their Causal Interrelations.* Retrieved from http://www.cogsci.rpi.edu/CSjarchive/procceding/2005/docs/p1955.pdf

Schwarzenberger, R.L.E& Tall, D.O. (1978).*Conflicts in the learning of real numbers and limits*, **Mathematics Teaching** 82, 44-49.

Stump, D. (2002). *The Limit Concept*. Retrieved from http://www.pa.msu.edu/~Stump/Stump

Tall, D. (1980). *Mathematical Intuition, with Special Reference to Limiting Processes*. Proceedings of the Fourth international congress on Mathematics education. Berkeley: 170-181.

Tall, D. and Vinner, S. (1981). *Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity*. **Educational Studies in Mathematics**, 12:151-169.